Theory of Arches and Suspension Bridges

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CHICAGO
THE MYRON C. CLARK PUBLISHING CO.
LONDON
E. & F. N. SPON, LTD., 57 Haymarket
1913

TRANSLATOR'S PREFACE.

STIMULATED by a profound admiration for the work of Professor Melan and prompted by a desire to fill a recognized gap in American engineering literature, the writer has undertaken the translation of this book on the Theory of Arches and Suspension Bridges. No other work of the same scope was to be found in any language, at least none that could compare with Melan's for thoroughness of treatment and masterly style. The place it has been given in the offices of consulting engineers and bridge departments, the frequent references to it in our technical literature and the employment of its formulæ or methods in the design of some of our largest structures indicate that, even before translation, there has been a real demand for Melan's book among American engineers. The work has been enthusiastically received in Europe where it has already gone through three editions and the highest honors have been awarded the author. In order to widen the sphere of usefulness of the book and to render it accessible to the entire profession in this country, the writer has been encouraged to give the time and effort required for this translation.

Professor Melan's book is here faithfully reproduced without any omissions. An appendix devoted especially to the design of masonry and reinforced concrete arches and a very exhaustive bibliography on arches and suspension bridges are included. The translator has carefully checked the derivation of all the formulæ and has corrected the few typographical errors found in the original. In some of the examples and tables the quantities have been converted from metric to English units. The notation has been altered wherever necessary to eliminate the German characters and abbreviations and to conform more closely to our standard symbols, and corresponding changes have been made in the figures and plates. In a few places brief explanatory notes have been inserted to help the American reader who, on account of the difference in training, may not be as familiar with the analytical and graphical devices employed or as agile in passing from formula to formula as the German student. With the exception of the above minor changes the original work has been kept intact.

In addition to its use as a work of reference by those engaged in the design of higher structures, the writer believes that this volume will be found admirably adapted as a textbook for advanced or post-graduate students in structural engi-

neering or applied mathematics and, from personal experience, he strongly recommends it for this purpose. Used in this way, the book will prove a liberal education in itself, rendering clear the fundamental principles of the analysis of structures, familiarizing the student with many helpful analytical and graphical devices of general application, training him in the independent derivation of formulæ and preparing him to do original work in structural theory and design.

The writer wishes to thank his friends in the engineering profession who have encouraged him to undertake this task, particularly Professor W. H. Burr and Dr. Myron S. Falk.

D. B. S.

University of Idaho, November 1, 1912.

FOREWORD

BY

HOFRAT J. MELAN.

THE present work, produced in English with the consent of the author, comprises the *Theory of Arches and Suspension Bridges* as it appears in the third enlarged edition of the "*Handbuch der Ingenieurwissenschaften,*" Volume 2, Part 5, 1906.

The design of arches and suspension structures is here thoroughly and exhaustively developed, both the analytical and graphic procedures being given in conformity with modern practice. All types of arches and suspension bridges, of any practical importance, are considered. The principles of the exact theory, taking into account the deformations produced by the loading in the case of lightly stiffened structures, are also developed (§ 9 and § 28).

The author can but express his hearty satisfaction and pleasure at this undertaking by a professional colleague to translate his work into English and thus render it accessible to the engineers of America. He hopes that the work will be found adapted for use in teaching this branch of bridge design and that the American bridge engineers, famous for the magnitude and skill of their achievements, will judge it favorably and find it useful for reference in theoretical problems.

Prague, Sept. 20, 1912.

J Melan

k. k. Hofrat, Professor des Brückenbaues an der deutschen technischen Hochschule in Prag.

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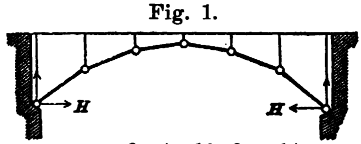
Theory of Arches and Suspension Bridges.

§ 1. Introduction.

Under the designation of arches and suspension bridges may be grouped all those forms of construction having the characteristic of transmitting oblique forces to the abutments even when the applied loads are vertical in direction. Each of these forces may be resolved into a vertical and a horizontal component; the direction of the latter component, either toward or away from the interior of the span, thus representing tension or pressure on the abutments, constitutes the criterion for distinguishing between suspension and arch constructions. In the former there is produced a *horizontal tension,* in the latter a *horizontal thrust*; and this horizontal force is either taken up directly by the end supports or else transmitted through a series of similar spans to the end of the construction where it is resisted by the stability of the supporting bodies. There should also be included those types of construction, based on the same fundamental principle, which, however, do not transfer the horizontal force to the abutments but take it up within the structure itself

Fig. 1.



$s+a=6+4=10,\ 2k=14.$

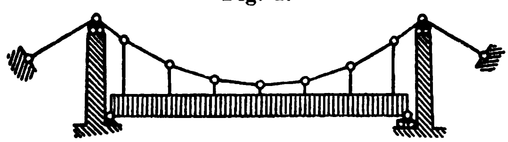
by means of a tie or strut joining the ends of the span. Finally, mention may be made of the systems resulting from the combination of the arch with the suspension bridge.

In a truss, external loading produces bending moments, and consequently stresses of both tension and compression will occur; in a suspension system, on the other hand, the principal parts are subjected exclusively to tensile stress, and in an arch the principal stresses are those of compression.

The starting point for the systems under consideration may be sought in the flexible cord or polygonal frame, Fig. 1, which may be viewed either as an arch or suspension bridge according as its concave side is directed downward or upward. Such a funicular frame has the characteristic property of altering its position of equilibrium with any change in the distribution of the loading, so that the form of the polygon will depend upon the applied loads. A polygon composed of s bars connected by

hinges will not constitute a rigid system if $s > 2$, but will be subject to deformation under moving loads. There are $s + 4$ unknown quantities, consisting of the stresses in the bars and the components of the end reactions; while the requirements of static equilibrium at the $k = s + 1$ panel points furnish $2\ (s + 1)$ equations of condition. Consequently, for $s > 2$, i. e., for a polygonal frame containing more than two bars, the number of conditional equations exceeds the number of unknowns. The superfluous equations can not, in general, be satisfied except by special relations between the constants, corresponding to a definite adjustment of the geometric form of the system to each particular disposition of the loading. This adjustment of form to

Fig. 2.

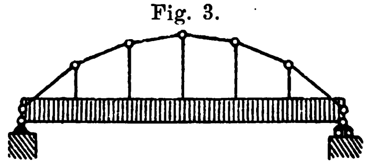


$s + a = 17 + 5 = 22, \ 3S + 2k = 3 \times 1 + 2 \times 9 = 21.$

varying loads can actually occur only in the case of the suspended polygon; while the arch-polygon can merely assume a position of unstable equilibrium, and will collapse upon the first change in the loading on account of the altered configuration required for equilibrium.

Arch-polygons, by themselves, are therefore unsuitable for use as bridges; they require, in addition, some type of stiffening construction to preserve their form, thereby converting them into rigid systems. A similar arrangement also becomes necessary in suspension bridges if it is desired to reduce the deformations of these structures. In the unstiffened suspension

Fig. 3.

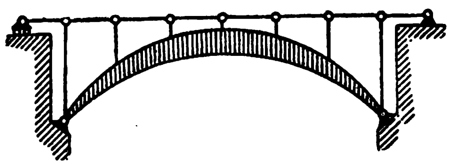


$s + a = 11 + 3 = 14, \ 3S + 2k = 3 \times 1 + 2 \times 5 = 13.$

bridges and arches, with loads applied at the panel points of the funicular polygon, the only stresses will be those of pure tension or compression, respectively; in the stiffened structures, however, subjected to changing load, bending stresses will also appear. The stiffening construction for flexible arches or suspension systems may consist in their connection with a straight truss (Figs. 2 and 3) whose elastic deflections will limit, in magnitude, the static deformations of the funicular polygon.

Instead of using this construction, the arch or suspension polygon may be made rigid in itself and thereby rendered capable of supporting any loading, even if the resultant forces do not coincide with the axes of the structural parts. Such a rigid system may be conceived as derived from the funicular polygon

Fig. 4.

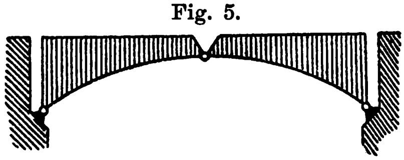


$s+a=15+7=22,\ 3S+2k=3\times1+2\times9=21.$

by replacing the hinged joints between the successive bars by rigid connections, and giving these bars such sections as will enable them to resist bending moments. The funicular polygon (Fig. 1) is thus replaced by a single curved rib, which may consist of either a girder or truss construction (Fig. 4).

Since the polygonal frame with hinged ends, hence with four reaction unknowns, constitutes a non-deformable construction when the number of bars is reduced to two, the union of

Fig. 5.



$a+2G=4+2\times1=6,\ 3S+2k=3\times2+0=6.$

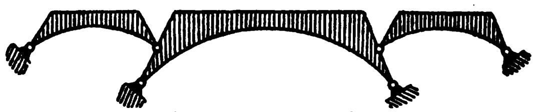
two ribs through an intermediate hinge likewise produces a stable bridge structure (Fig. 5). To this system other parts may be linked, according to the number of new reaction unknowns thereby contributed.

We thus derive various types of stable bridges (Figs. 5 to 8) which are classed as suspension bridges or arches whenever, as specified at the outset, they produce oblique reactions at the ends of the principal curved or polygonal portion of the structure; these reactions may be taken up either by the abutments or by a tie-rod construction (Fig. 3).

For determining the stresses in these structures, the equations of static equilibrium may or may not suffice. In the former case we have *statically determinate* systems, in the latter *statically indeterminate*. This distinction is to be drawn particularly with reference to the unknown external forces; any

bridge type can easily be classified as to its external determinateness or indeterminateness if we conceive it as built up of bars, or of bars and ribs, and compare the number of unknown forces with the number of independent equations of condition available. The pivoted connection of two ribs is called a hinge, and the similar connection of two or more bars is called a panel-point. Each rib furnishes *three* equations of equilibrium for its applied forces, and each panel-point furnishes *two* such equations. The unknowns include first the reaction components whose number is given by the rule: *one*

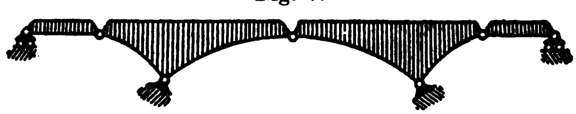
Fig. 6.



$a+2G=8+2\times 2=12,\ 3S=9.$

unknown for each free end, i. e., each end capable of free motion on sliding plates or rollers; *two* for each hinged end, i. e., each end capable of rotation but not of translation; and *three* for each fixed end, rigidly anchored. Similarly each intermediate hinge represents two unknowns and each intermediate sliding joint one; and the stresses in the bars are additional unknowns. Therefore, if a structure consists of S ribs with G hinges and of s bars with k panel-points, and if a is the number of reaction unknowns, the total number of unknowns is $a + s + 2\ G$ and the number of corresponding equations of equilibrium is

Fig. 7.



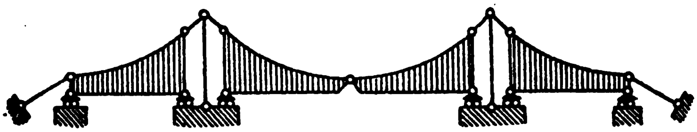
$a+2G=6+2\times 3=12,\ 3S=3\times 4=12.$

$3\ S + 2\ k$. For all combinations in which the external forces are statically determinate, $a + s + 2\ G = 3\ S + 2\ k$; and for those statically indeterminate, $a + s + 2\ G > 3\ S + 2\ k$.

We thus find that such systems as shown in Figures 2 and 3 belong to the statically indeterminate class, but that they may be rendered statically determinate with reference to the forces affecting the component structures (polygon and stiffening truss) by introducing an intermediate hinge in the stiffening truss. The same applies to the arch rib with end-hinges (Fig. 4), which also becomes statically determinate when a center-hinge is introduced, making it a three-hinged arch (Fig. 5). In this case, however, the same result may be obtained by using some

weighting device to keep the horizontal component of the end-reactions at a constant value; or by changing one of the end-hinges to a free end (sliding on an inclined plane) so as to diminish the number of reaction unknowns by one (Fig. 9). The corresponding reaction will then act constantly normal to the plane of the rollers, so that the reaction at the other end may be determined by simple resolution of forces. This arrange-

Fig. 8.



$a+s+2G=6+8+2=16,\ 3S+2k=3\times 4+2\times 2=16.$

ment, however, is not classed among true arches, but is considered simply a girder whose expansion end slides on inclined bearings.

In general, subdividing any rib by the insertion of a hinge will diminish the relative number of unknowns by one. Thus the arrangement in Fig. 6, which is *three-fold* statically indeterminate, may be rendered determinate by introducing *three* hinges. On the other hand, the statically determinate structure in Fig. 7 may be rendered doubly indeterminate by omitting the hinges in the two side-spans.

Not all combinations which may thus be formed, however, are practical for construction. Notwithstanding the static

Fig. 9.

determinateness or even indeterminateness of the entire combination, there may be so great a degree of mobility in the individual parts or in the structure as a whole that its rigidity may be inadequate for practical requirements. This will be the case in the three-hinged arch (Fig. 5), for example, if the two end-hinges and the center-hinge lie nearly or exactly in a straight line; or, in general, whenever any one of the component parts of the structure is unrestrained or imperfectly restrained from rotation (or translation = rotation about an infinitely distant point).

To make this clear, it is necessary to consider the structure as a kinematic chain and to enlarge the concept of the hinge.

In a plane rib $a\ b\ c$ (Fig. 10), conceived as a kinematic element, let two points a and b be constrained to move along definite paths. The intersection o of the normals to these paths constitutes the instantaneous center of rotation, having the effect of a fixed pivot. The rib will not be restrained from rotating about this point o unless a third point c in the rib is either fixed or constrained to move along a path whose normal does not pass

Fig. 10.

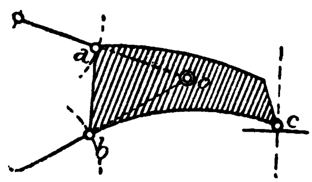
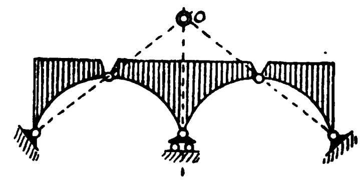
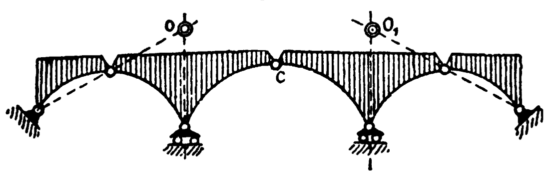


Fig. 11.



through o. This requirement is not satisfied, for example, in the case of the middle rib of Fig. 11, since it has but a single instantaneous center o; this arrangement, therefore, although satisfying the general criterion for static determinateness, is not stable and cannot be used for a bridge. It is similarly apparent that the arrangement in Fig. 12 is not stable if the center-hinge c lies in the line joining the two centers $o\ o_1$; the

Fig. 12.



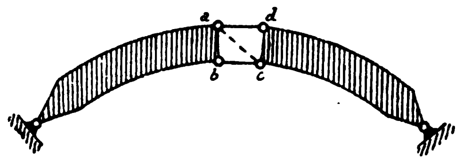
closer c comes to the line $o\ o_1$, the less is the rigidity of the structure and the greater will be the deformations.

In accordance with the generalized concept just developed, the hinge-joint between two ribs may be replaced by a pair of pin-ended links forming a "hinged quadrilateral." The intersection of the axes of the links determines the center of rotation, i. e., the virtual pivot taking the place of the fixed hinge. Thus the arrangement in Fig. 13 represents a statically determinate three-hinged arch; but it will not be a stable system if the two links $a\ d$ and $b\ c$ intersect in the line joining the two end-hinges or are parallel to this line. By the insertion of another bar in the "hinged quadrilateral" (e. g., $a\ c$ in Fig. 13), the hinge effect is suspended and the structure Fig. 13 is changed to a singly indeterminate two-hinged arch.

The end-hinges, also, may be replaced in their action by pairs of links (Fig. 14). By adding a third link, rotation is prevented and the ends become anchored. Thus with the bars *b c* and *f g* provided, Fig. 14 represents a hingeless arch and the degree of indetermination is raised to three.

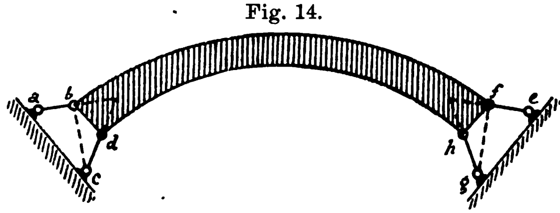
The individual ribs composing the structure may either be

Fig. 13.



made solid, i. e., of plate-girder form, or framed of separate members like a truss. If the *external* forces acting on each rib are known, then the *internal* forces, i. e., the stresses in the different members of the rib, can be determined. For this purpose the methods of pure statics will suffice in the case of

Fig. 14.



statically determinate truss systems, but for indeterminate forms, including solid plate girders, the elastic deformations must be taken into account. For criteria of static determinateness or indeterminateness of trusses, the reader is referred to other works on the subject.

§ 2. **General Method of Design.** The fundamental principles of design of ordinary trusses or girders also apply to arches and suspension bridges. As in the simpler structures, it is necessary to first determine the external forces or reactions produced by a specified loading. Knowing these, we can proceed to evaluate the stresses within the structure. For the solid-web portions, we apply the general theory of flexure of curved ribs which, as an approximation, may be reduced to the simpler theory of straight beams; for the open-web or framed portions, the ordinary rules and methods of truss design are applicable. These methods, as is well known, assume an ideal framework in which the external forces are applied and the members meet one another only at their end-points, and all necessary rotation about these points may take place without any resistance,—assumptions which, as a rule, are not fulfilled in structures as built, rendering it necessary to estimate the magnitude of the deviations or so-called secondary stresses. It should also be noted here that the results of the theory deduced for solid ribs may often be extended to structures with latticed or trussed webs.

If the structure is statically determinate with reference to the external forces, the ordinary rules for the composition and resolution of coplanar forces suffice to definitely determine the end reactions and, in determinate truss systems, the stresses in the members as well. In statically indeterminate arrangements, the missing equations of condition must be deduced from the displacements of the points of application of the external forces, i. e., from the elastic deformations of the structure. To establish these equations, we may use the "Theorem of Virtual Displacements," first applied to the design of indeterminate structures by *Mohr*. The same equations may also be derived, and sometimes more conveniently, by applying the "Theorem of Least Work" established by *Castigliano* and *Frankel: In any elastic system in a condition of equilibrium, such stresses must appear as will make the total work of deformation a minimum.* Consequently, if this work of deformation is expressed as a function of the indeterminate stresses or reactions, the differential coefficient of the function with respect to these unknowns, equated to zero as required for a minimum, will give the equations of condition necessary for the determination of these unknowns.

In place of the analytical treatment, most of the problems of this class may also be solved by graphic methods. For statically indeterminate structures, the graphic process consists in drawing one or more deformation polygons (*Williot's* dis-

placement diagram or deflection polygon), enabling the indeterminate quantities to be found. But even if the latter are determined analytically, graphic methods may still be used to advantage for arches and suspension bridges, especially in determining the internal stresses.

There remains to be noted that in evaluating the work of deformation or in drawing the displacement diagram, the operations may be simplified within the limits of permissible approximation by neglecting those elements, as the web members of a truss or the shears in a girder, which have a negligible influence on the total work of deformation.

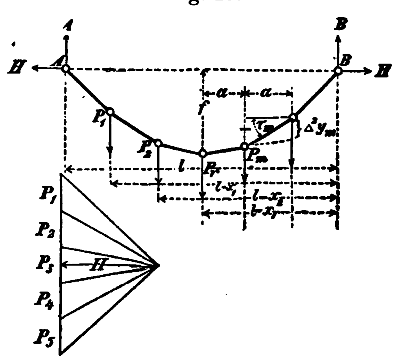
A. The Flexible Arch and the Unstiffened Cable.

§3. The Funicular Polygon for Given Vertical Loading. If concentrated vertical loads are applied on a cord, fastened at its ends and considered weightless, it will assume a definite polygonal form dependent upon the relations between the loads. If A denotes the vertical component of the tension at one of the supports, and if $P_1 \ldots P_m$ are the applied loads as far as the m^{th} side of the polygon, the vertical component of the stress in this side will be

$$V_m = A - P_1 - P_2 \ldots - P_m,$$

while the horizontal component H will be the same for all the

Fig. 15.



sides. The angle τ_m between the m^{th} side and the horizontal is given by

$$\tan \tau_m = \frac{V_m}{H} = \frac{A - P_1 - P_2 \ldots - P_m}{H} \quad \ldots\ldots\ldots (1.$$

and the tension in the cord by

$$T_m = \sqrt{V_m^2 + H^2} = H \sec \tau_m \quad \ldots\ldots\ldots\ldots\ldots\ldots (2.$$

If x_m, y_m and x_{m+1}, y_{m+1} denote the coordinates of the vertices adjoining the m^{th} side, referred to the point A (Fig. 15) as origin, we also have

$$y_{m+1} - y_m = \frac{A - P_1 - P_2 \ldots - P_m}{H} \cdot (x_{m+1} - x_m) \ldots . (3.$$

or $$\Delta y_m = \frac{A - P_1 \ldots - P_{m-1} - P_m}{H} \cdot \Delta x_m.$$

In like manner we obtain for the side preceding

$$\Delta y_{m-1} = \frac{A - P_1 - P_2 \ldots - P_{m-1}}{H} \cdot \Delta x_{m-1}.$$

If we write

$$\Delta y_m - \Delta y_{m-1} = y_{m+1} - 2y_m + y_{m-1} = \Delta^2 y_m$$

and if $\Delta x_m = \Delta x_{m-1} = a$, i. e., if the horizontal spacing of the vertices of the polygon is uniform, then we obtain

$$\Delta^2 y_m = -\frac{P_m}{H} \cdot a \ldots\ldots \quad (3^a.$$

In general we have

$$(\tan \tau_m - \tan \tau_{m-1}) = -\frac{P_m}{H} \ldots\ldots \quad (3^b.$$

The funicular polygon for the loads P and the horizontal tension H may also be determined graphically in the well-known manner (Fig. 15).

If the points of suspension of the cord lie in a horizontal line, the force A is the same as the reaction of a simply supported beam and is found from the moment of the external loads:

$$A = \frac{\Sigma P\,(l - x)}{l} \ldots\ldots \quad (4.$$

In this case, if f is the versine of the funicular polygon and r the lowest vertex, we have, from the equation of moments,

$$H = \frac{M_r}{f} = \frac{Ax_r - P_1\,(x_r - x_1) - P_2\,(x_r - x_2) - \ldots P_{r-1}\,(x_r - x_{r-1})}{f} \ldots \quad (5.$$

If the versine f is not given, but the length of the cord instead; and if, in addition, the points of attachment of the suspended loads 1, 2 .. and, thereby, the lengths $e_0 e_1 e_2$ .. of the individual sides of the polygon are fixed, then the horizontal tension is given by the relation:

$$\frac{e_0}{\sqrt{H^2 + A^2}} + \frac{e_1}{\sqrt{H^2 + (A - P_1)^2}} + \frac{e_2}{\sqrt{H^2 + (A - P_1 - P_2)^2}} + \ldots = \frac{l}{H} \ldots \quad (6.$$

It is readily seen that the stresses T in the successive members of the polygon increase toward the points of support and attain their maximum values in the first and last members of the system; also that for downward acting loads the stresses in a polygon convex downward will be purely those of *tension,* and in a polygon convex upward they will be purely those of *compression.*

If the loads are continuously distributed, the funicular polygon becomes a continuous curve. If q is the load per

horizontal linear unit at any point having the abscissa x, equ. (3) becomes

$$dy = \frac{A - \int q\,dx}{H} \cdot dx,$$

from which (by differentiation) we obtain the following as the differential equation of the equilibrium curve:

$$H \frac{d^2y}{dx^2} = -q \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (7.$$

If r is the radius of curvature of the equilibrium curve, $\frac{d^2y}{dx^2} = -\frac{1}{r} \sec^3\tau$. Substituting this in the last equation (7), we obtain, with the aid of equ. (2),

$$T_x = q \cdot r \cdot \cos^2 \tau \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8.$$

and, for a load q_0 and radius of curvature r_0 at the crown of the curve,

$$H = q_0 r_0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (9.$$

For a *uniformly* distributed load, i. e., for a constant q, if we take the origin of coordinates at the crown, the integration of equation (7) will give

$$y = \frac{q x^2}{2 H} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (10.$$

Hence the equilibrium curve will, in this case, be a parabola. If l is the span and f the rise (versine), then

$$H = \frac{q l^2}{8 f}; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (11.$$

consequently, by equ. (10),

$$y = 4 f \frac{x^2}{l^2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (12.$$

The largest stress in the cord will be

$$T_{\max} = \sqrt{H^2 + \left(\frac{1}{2} q l\right)^2} = \frac{q l^2}{8 f} \sqrt{1 + \left(\frac{4 f}{l}\right)^2} \ldots \quad (13.$$

If the load is not constant per horizontal unit, but per unit length of the cord, then the equilibrium curve takes the form of the common catenary. In that case, with the origin of co-ordinates at the crown of the curve, equ. (7) becomes

$$H \frac{d^2 y}{d x^2} = -g \sec \tau = -g \left(1 + \left(\frac{d y}{d x}\right)^2\right)^{1/2}.$$

From this, with $\frac{1}{c} = \frac{H}{g}$ as the parameter of the catenary, we obtain the following equation for the curve:

$$y = \frac{1}{2c}\left(e^{cx} + e^{-cx} - 2\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots (14.$$

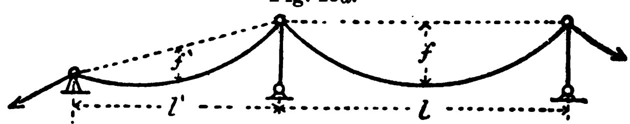
The cord length s of the catenary is given by

$$s = \frac{1}{c}\sqrt{2cy + c^2 y^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (15.$$

The foregoing equations for H apply also to an obliquely suspended cord if the ordinates y are measured from the arch-chord or line joining the points of support.

If the funicular polygon is carried over an intermediate supporting pier (Fig. 15a) in such a manner as to leave it free either

Fig. 15a.



to slide itself or to displace the point of support, then the cord will assume a position of equilibrium yielding the same value of H on both sides of the pier. If the load is uniformly distributed per horizontal unit of length, with the intensity q in one span and q' in the other, the equalizing of the horizontal tensions given by equ. (11) will impose the following relation between the versines of the two parabolas:

$$\frac{f'}{f} = \frac{q' l'^2}{q l^2}.$$

§4. The Unstiffened Suspension Bridge.

1. Form of the Cable and Value of the Horizontal Tension. Let q_0 denote the weight per horizontal linear unit of the roadway suspended from the cable together with the assumed uniformly distributed live load.

a.) *Chain-cable bridges.* We make the assumption, near enough to the truth for all practical conditions, that the weight of the chain is so distributed as if its curve were a parabola and its cross-section at every point proportional to the stresses under maximum loading. Let g_0 be the weight per unit length of the chain at the crown; then, at a distance x from the crown, the weight of the chain per horizontal linear unit will be

$$g_x = g_0 \sec^2 \tau.$$

Substituting the following value for the parabola of equ. (12),

$$\sec^2 \tau = 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{64 f^2 x^2}{l^4},$$

there results

$$g_x = g_0 \left(1 + \frac{64 f^2 x^2}{l^4}\right).$$

Let the weight of the suspension rods per horizontal linear unit

$$= j\, y = j\, \frac{4 f x^2}{l^2}.$$

Then, transferring the origin of coordinates to the crown of the curve, equ. (7) becomes,

$$H \frac{d^2 y}{d x^2} = q_0 + g_0 + \left(g_0 \frac{64 f^2}{l^4} + j \frac{4 f}{l^2}\right) x^2.$$

Introducing the abbreviations

$$\left.\begin{aligned} & q_0 + g_0 = q \\ & \frac{2}{3} \frac{f}{l^2}\left(16 \frac{f}{l^2} g_0 + j\right) = r, \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (16.$$

then the double integration of the above differential equation will yield

$$y = \frac{x^2}{2H}(q + r x^2) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (17.$$

Substituting in this equation of the curve the coordinates of one of its points, namely $x = \frac{l}{2}$ and $y = f$, we obtain

$$H = \frac{l^2}{8f}\left(q + r\frac{l^2}{4}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (18.$$

Replacing the value of r from equ. (16), we have the following expression for the horizontal tension:

$$H = g_0\left(1 + \frac{8}{3}\frac{f^2}{l^2}\right)\frac{l^2}{8f} + \left(q_0 + \frac{f}{6}j\right)\frac{l^2}{8f} \ldots\ldots \quad (18^a.$$

If A_0 is the cross-section of the chain at the crown in sq. in., γ its specific weight in lbs. per cu. ft., s its stress under severest loading (in lbs. per sq. in.), then $g_0 = C.A_0.\frac{\gamma}{144}$ (lbs. p. l. f.) and $A_0 = \frac{H}{s}$. Here C is a coefficient representing the increase in weight allowed for the details of construction such as eye-bar heads, pins, etc., and may be assumed equal to 1.15 to 1.20. Substituting the above relations in equ. (18^a) and solving for A_0, we find

$$A_0 = \frac{l^2}{8f}\frac{q_0 + \frac{f}{6}j}{s - \frac{C.\gamma}{144}\left(\frac{l^2}{8f} + \frac{1}{3}f\right)} \ldots\ldots\ldots\ldots \quad (19.$$

The weight of the semi-cable will be $G = \int_0^{\frac{l}{2}} g_x . dx$, or

$$G = C.\frac{A_0.\gamma}{144}\frac{l}{2}\left(1 + \frac{16}{3}\frac{f^2}{l^2}\right) \ldots\ldots\ldots\ldots\ldots \quad (20.$$

The vertical component of the end-reaction will then be

$$V = \frac{1}{2}q_0 l + \frac{1}{6}j.f.l + G \ldots\ldots\ldots\ldots\ldots\ldots \quad (21.$$

and the greatest tension in the chain is given by

$$T_{max} = \sqrt{V^2 + H^2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (22.$$

The cross-section of the chain at the points of support must therefore be

$$A_{max} = \frac{T_{max}}{s} = A_0\sqrt{1 + \frac{V^2}{H^2}} \ldots\ldots\ldots\ldots\ldots \quad (22^a.$$

At these points, the angle which the cable makes with the horizontal is determined by $tan\ \tau = \frac{V}{H}$ or, approximately,

$$tan\ \tau = \frac{4f}{l}\left(1 + \frac{g_0}{g_0 + q_0}\frac{8}{3}\frac{f^2}{l^2}\right) \ldots\ldots\ldots\ldots \quad (23.$$

b.) *Wire-cable bridges.* A wire cable has a uniform cross-

section throughout. If g is the weight of the cable per linear unit (p. l. f.), the horizontal projection of this weight will give

$$g_x = g \,.\, sec\, \tau$$

or, assuming a parabolic curve for the cable,

$$g_x = g \,.\, sec\, \tau = g \sqrt{1 + \frac{64\, f^2 x^2}{l^4}}\,.$$

The weight of the suspension rods will be assumed as above. On integrating the differential equation of the equilibrium curve, we obtain

$$y = \frac{x^2}{2\,H}\,(g + q_0 + r\,x^2) \ldots\ldots\ldots\ldots \quad (24.$$

where

$$r = \frac{2}{3}\,\frac{f}{l^2}\left(8\,\frac{f}{l^2}\,g + j\right) \ldots\ldots\ldots\ldots \quad (25.$$

The horizontal tension will be

$$H = g\,\left(1 + \frac{4}{3}\,\frac{f^2}{l^2}\right)\frac{l^2}{8\,f} + \left(q_0 + \frac{f}{6}\,j\right)\frac{l^2}{8\,f} \ldots\ldots \quad (26.$$

The weight of the half-cable will be approximately

$$G = g\,\frac{l}{2}\left(1 + \frac{8}{3}\,\frac{f^2}{l^2}\right) \ldots\ldots\ldots\ldots \quad (27.$$

The values of V and T_{max} are given by formulae (21) and (22). The angle which the cable at the tower makes with the horizontal is obtained from $tan\, \tau = \frac{V}{H}$, or approximately,

$$tan\ \tau = \frac{4\,f}{l}\left(1 + \frac{g}{g + q_0}\,\frac{4}{3}\,\frac{f^2}{l^2}\right) \ldots\ldots\ldots \quad (28.$$

The constant cross-section of the cable is determined by $A = \frac{T_{max}}{s} = \frac{H}{s}\, sec\, \tau$. With the value of H from equ. (26) and with $g = \frac{\gamma\, A}{144}$, there results

$$A = \frac{\left(q_0 + \frac{f}{6}\,j\right)\frac{l^2}{8\,f} sec\, \tau}{s - \frac{\gamma}{144}\left(\frac{l^2}{8\,f} + \frac{1}{6}\,f\right) sec\, \tau} \ldots\ldots\ldots \quad (29.$$

The *length* of the chain or cable will be given by

$$\left.\begin{aligned} L &= 2\int_0^{\frac{l}{2}} \sqrt{1 + \left(\frac{d\,y}{d\,x}\right)^2}\, d\,x \\ &= 2\left\{ x + \frac{1}{\frac{d^2 y}{d\,x^2}}\left(\frac{d\,y}{d\,x}\right)^3\left[\frac{1}{6} - \frac{1}{40}\left(\frac{d\,y}{d\,x}\right)^2 + \ldots\right]\right\}_0^{\frac{l}{2}}, \end{aligned}\right\}$$

where $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ are to be calculated from equ. (17). Substituting their values, we obtain

$$L = l\left\{1 + \frac{16f^2}{l^2}\frac{\left(q + r\frac{l^2}{2}\right)^2}{\left(q + r\frac{l^2}{4}\right)^2\left(q + \frac{3}{2}rl^2\right)} \cdot \left[\frac{1}{6} - \frac{2f^2}{5l^2}\left(\frac{q + r\frac{l^2}{2}}{q + r\frac{l^2}{4}}\right)^2 + \ldots\right]\right\} \quad \ldots (30.$$

For small values of r, i. e., for a small ratio of rise to span, we may write the following expression for the cable-length, applicable also to flat parabolic curves:

$$L = l\left\{1 + \frac{8}{3}\frac{f^2}{l^2} - \frac{32}{5}\frac{f^4}{l^4}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (31.$$

2. Maximum Attainable Span. The theoretical limit of span attainable with a suspension bridge is determined by the condition that the cable shall have a finite cross-section, hence that the denominator of equ. (19) or (29) must remain positive. This criterion becomes

for chain-cables,

$$l_{max} \overline{\overline{<}} \frac{1152\frac{s}{C\gamma}}{\frac{l}{f} + \frac{8}{3}\frac{f}{l}}$$

for wire-cables,

$$l_{max} \overline{\overline{<}} \frac{1152\frac{s}{\gamma}}{\left(\frac{l}{f} + \frac{4}{3}\frac{f}{l}\right)\sqrt{1 + 16\frac{f^2}{l^2}}} \quad \ldots\ldots\ldots (32.$$

We thus obtain the following theoretic values of the maximum spans for the given working stresses and rise-ratios:

	$\frac{f}{l} =$		$\frac{1}{8}$	$\frac{1}{10}$	$\frac{1}{12}$	$\frac{1}{14}$	$\frac{1}{16}$
	lbs./ft.^{3}	lbs./in.^{2}	ft.	ft.	ft.	ft.	ft.
Chain-Cables....	$C\gamma = 560$	$s = 12000$	2960	2403	2020	1740	1527
		$s = 18000$	4440	3605	3030	2610	2290
Wire-Cables.....	$\gamma = 490$	$s = 45000$	11580	9700	8290	7220	6380
		$s = 60000$	15440	12933	11053	9627	8507

In the above spans, if the cable is to have a finite cross-section, it must carry no other load than its own dead weight. The above solution, consequently, does not give the maximum practicable span for suspension bridges which have to carry a definite useful load in addition to the weight of the cable. We may, however, calculate the span l for a cable of the maximum practicable cross-section when the rise-ratio, working stress and applied load

are specified. If the weight of the cable per unit length $= g$, and if we put $g' = g\left(1 + \frac{4}{3}\frac{f^2}{l^2}\right)$, also if q_0 denotes the suspended load p. l. f., then we have

$$l = l_{\max} \cdot \frac{g'}{g' + q_0} \quad \text{(33.}$$

It is important to note here that q_0 is not quite independent of the span-length, but increases with it on account of the weight of the stiffening construction and the wind-bracing.

Note:—The problem of the maximum practicable span for suspension bridges, arising in connection with the projects for bridging the North River, was investigated by a special commission of U. S. army officers. Assuming 16 cables, each containing 6000 steel wires, No. 3, B. W. G., with a total cross-section of 5058 sq. in. and weighing $g = 17{,}200$ lbs. p. l. f.; also adopting a working stress of 60,000 lbs. per sq. in., a live load consisting of a six-track railway giving $p = \frac{27{,}540{,}000}{l}$ lbs. p. l. f. and a corresponding total load of

$$q_0 = 13605 + \frac{27{,}764{,}726}{l} + 3.24906\, l + 0.00055335\, l^2 + 0.000000003\, l^3,$$

there is obtained, for a rise-ratio of 1:8, the value of $l = 4{,}335$ ft. for the limiting span.—(Report of Board of Engineer Officers as to "the maximum length of span practicable for suspension bridges": Major Chas. W. Raymond, Captain W. H. Bixby, Captain Edward Burr; Washington, Government Printing Office, 1894.)

3. Economic Ratio of Rise to Span. The rise (or versine) of the cable affects, on the one hand, its own weight and that of the suspension rods and, on the other hand, the cost of the towers. If we overlook the effect on the backstays and on the masonry of the anchorage, we may determine approximately the rise-ratio which will make the total cost a minimum. Denoting this ratio by $\frac{f}{l} = n$, the ratio of the cost of the towers per foot of height to the price per pound of steel construction by P, and using the abbreviation $\frac{C\gamma}{144\, s} = \epsilon$, then, with the aid of equ. (20) and a small admissible simplification, there is obtained the equation of condition

$$\frac{\epsilon\left(1 + \frac{16}{3} n^2\right)}{8\frac{n}{l} - \epsilon\left(1 + \frac{8}{3} n^2\right)}\, q_0 l + \frac{1}{3}\epsilon\, n\, q_0 l^2 + P n l = min.$$

Differentiating and solving for n, and, for abbreviation, putting the very small quantity

$$\frac{2\epsilon q_0 l + 3P}{\epsilon q_0 l + P} \cdot \frac{\epsilon l}{12} = a, \text{we obtain,}$$

$$n = a + \sqrt{a^2 + \frac{1}{8}\frac{\epsilon q_0 l}{\epsilon q_0 l + P}} \quad \text{(34.}$$

Neglecting the value of α, we have,

$$n=\sqrt{\frac{1}{8}\frac{\epsilon q_0 l}{\epsilon q_0 l+P}} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(35.}$$

For example, if $P=1650$, $\epsilon=0.0003$, $q_0=2000$ lbs. p. l. f. and $l=300$ ft., then

$$n=0.11=\frac{1}{9}.$$

In order, however, to reduce the deflections as far as practicable (see following paragraph), it may be desirable to reduce this value of the best rise-ratio.

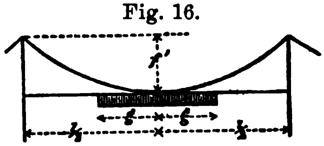
4. Deformations. In order to simplify the following investigations, we will assume the cables when unloaded, i. e., subjected merely to their own weight, to conform to parabolic curves; this is very closely true with small ratios of rise to span. We will also neglect the resistance to deformation afforded by the friction at the pins of the chain or by the stiffness of the wire-cable. Let

$g=$ the total dead weight of the bridge, p. l. f.

$p=$ the uniformly distributed live load, p. l. f.

a.) *Maximum crown-deflection produced by deformation under load.* This will occur when a certain central portion of length 2ξ is loaded. For the present we will not consider the possible displacements of the cable-saddles on the towers, but will assume the cables as fixed at the ends. Let f' denote the versine of the cable when loaded, so that by equ. (5),

Fig. 16.



$$f'=\frac{1}{8}\frac{gl^2}{H}+\frac{1}{2}\frac{p\xi(l-\xi)}{H} \quad \ldots \text{(a.}$$

We next equate the expressions for the cable-length corresponding to the unloaded and loaded conditions respectively, the former being obtained from the approximate equ. (31) by neglecting the third term therein; we thus obtain:

$$\frac{L}{2}=\frac{l}{2}\left(1+\frac{8}{3}\frac{f^2}{l^2}\right)=\frac{l}{2}+\frac{1}{H^2}\left\{\frac{1}{6}(p+g)^2\xi^3+\frac{1}{4}\left[p\xi+\frac{1}{4}g\,(l+2\xi)\right]^2(l-2\xi)+\frac{1}{24}g^2\left(\frac{l}{2}-\xi\right)^3\right\} \quad \ldots \text{(β.}$$

The solution of this equation yields

$$H=\frac{1}{4f}\sqrt{l\left\{\frac{1}{4}g^2l^3+\frac{3}{2}pgl^2\xi+3p^2l\xi^2-(4p^2+2pg)\xi^3\right\}} \quad \ldots \text{(γ.}$$

Substituting this value in equ. (α), there results,

$$\frac{f'}{f}=\frac{2p\xi(l-\xi)+\frac{1}{2}gl^2}{\sqrt{l\left\{\frac{1}{4}g^2l^3+\frac{3}{2}pgl^2\xi+3p^2l\xi^2-(4p^2+2pg)\xi^3\right\}}} \quad \text{(36.}$$

By differentiating this expression with reference to ξ, we obtain the following condition for a maximum value of f':

$$\left.\begin{array}{l} 2\left(\frac{\xi}{l}\right)^4\left(2\frac{p}{g}+1\right)\frac{p}{g}+2\left(\frac{\xi}{l}\right)^3\left(\frac{p}{g}-1\right)\frac{p}{g} \\ \qquad +\frac{3}{2}\left(\frac{\xi}{l}\right)^2\left(1-\frac{p}{g}\right)-\frac{\xi}{l}+\frac{1}{8}=0 \end{array}\right\} \quad (37.$$

Solving this equation for $\frac{\xi}{l}$ and substituting the result in equ. (36), we obtain the following values for the maximum crown deflection $\Delta f_1 = f' - f$:

For $\frac{p}{g}=$	0	$\frac{1}{2}$	1	2	3
$\frac{\xi}{l}=$	0.5	0.141	0.126	0.113	0.107
$\Delta f_1=$	0	0.028	0.045	0.067	0.079.f.

From this tabulation we may obtain the following approximate values, sufficiently accurate between the limits $\frac{p}{g}=\frac{1}{4}$ to 4:

$$\left.\begin{array}{l} \frac{\xi}{l}=0.1+0.025\frac{g}{p} \\ \Delta f_1=\left(0.007+0.046\frac{p}{g}-0.0075\frac{p^2}{g^2}\right)f \end{array}\right\} \quad \ldots (38.$$

b.) *Crown-deflection due to elongation of cable.* The differentiation of equ. (31) yields, on putting the rise-ratio $\frac{f}{l}=n$,

$$\Delta f_2 = \frac{15}{16\,(5\,n-24\,n^3)}\,\Delta L \quad \ldots\ldots (39.$$

The elongation of the cable (ΔL) may be due to its elastic strain, to temperature variation, or to a yielding of the anchorages.

If s is the intensity of stress in the cable under maximum live load ($=p$), then the elastic stretch due to dead load (g) will be

$$\Delta L_g = \frac{s}{E}\,\frac{g}{g+p}\,.\,L \quad \ldots\ldots (40.$$

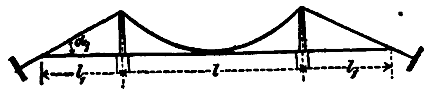
and the increase in stretch due to live load will be

$$\Delta L_p = \frac{s}{E}\,\frac{p}{g+p}\,.\,L \quad \ldots\ldots (41.$$

The elongation produced by a temperature variation of $\pm t^0$, with a coefficient of expansion of $\omega = .0000069$, is

$$\Delta L_t = \pm\,\omega\,t\,L \quad \ldots\ldots (42.$$

Fig. 17.



If the cable over the main span is continued on each side as a backstay to the anchorage (Fig. 17), and if the cable is capable of slipping over the fixed saddles, then the value

of L in equs. (39) to (42) must consist of the total length of cable between anchorages; consequently

$$L = l\left(1 + \frac{8}{3}n^2 - \frac{32}{5}n^4\right) + 2\,l_1 \sec \alpha_1 \ldots\ldots\ldots (43.$$

If, however, a displacement of the saddle will occur before the cable will slip, then with $\Delta L =$ the elongation of the main cable and $\Delta L_1 =$ the elongation of one of the backstays, we should have

$$\Delta f_2 = \frac{15}{16\,(5n - 24n^3)}\,\Delta L + \frac{2}{\sec \alpha_1}\,\frac{15 - 8\,(5n^2 - 36\,n^4)}{16\,(5\,n - 24\,n^3)}\,\Delta L_1.$$

Substituting the values

$$\Delta L = c\,.\,L = c\,.\left(1 + \frac{8}{3}n^2 - \frac{32}{5}n^4\right)l,$$

$$\Delta L_1 = c\,.\,L_1 = c\,.\,\sec \alpha_1\,.\,l_1,$$

where c is given by the coefficients of L in equs. (40) to (42), we obtain

$$\Delta f_2 = \left\{\frac{n}{2}\,l - nl_1 + \frac{15 + 96\,n^4}{16(5\,n - 24\,n^3)}\,(l + 2l_1)\right\}\cdot c \ldots (44.$$

c.) *Crown-deflection produced by displacement of the saddles.* If a displacement of the saddles reduces the effective span of the cable by an amount $\Delta\,l$, without a simultaneous change in the total length of the cable, there will result a sag or deflection at the crown amounting to

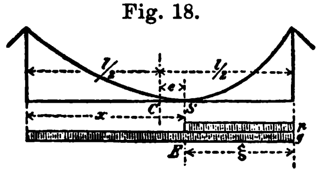
$$\Delta f_3 = \frac{15 - 8\,(5\,n^2 - 36\,n^4)}{16\,(5\,n - 24\,n^3)}\,\Delta\,l \ldots\ldots\ldots\ldots\ldots (45.$$

If the saddle-displacement is accompanied by a slipping of the cable, so that the total length of the latter between anchorages (Fig. 17) remains unchanged, then the crown deflection becomes

$$\Delta f_3 = \frac{15 - 8\,(5\,n^2 - 36\,n^4) - 15\,.\,\sec \alpha_1}{16\,(5\,n - 24\,n^3)}\,\Delta\,l \ldots\ldots (46.$$

d.) *Maximum horizontal displacement of the crown.* For the lowest point of the cable curve, we have $\frac{d\,M}{d\,x} = 0$; accordingly, with the notation of Fig. 18, so long as the crown (S) is to the left of the head of the load (E),

Fig. 18.



$$\frac{d\,M}{d\,x} = \frac{1}{2}\,g\,(l - 2\,x) + \frac{1}{2}\,p\,\frac{\xi^2}{l} = 0.$$

Consequently x will have its maximum value when ξ has its

maximum value, hence when $\xi = l - x$. Substituting this value in the above equation, we obtain

$$\frac{\xi}{l} = -\frac{g}{p} + \sqrt{\frac{g}{p} + \frac{g^2}{p^2}}.$$

Hence the maximum deviation of the crown from the center of the span will be

$$\frac{e}{l} = \frac{1}{2} + \frac{g}{p} - \sqrt{\frac{g}{p} + \frac{g^2}{p^2}} \quad \ldots\ldots\ldots\ldots\ldots (47.$$

The total rise of the cable may be assumed invariable for all ordinary values of $g : p$. Consequently, the uplift of the cable at the center of the span will amount to

$$\Delta f_4 = \left(\frac{2e}{l + 2e}\right)^2 . f \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (48.$$

We thus obtain the following values:

for $\frac{g}{p} =$	$\frac{1}{3}$	$\frac{1}{2}$	1	2	3
$e =$	0.167	0.134	0.086	0.051	0.036 l
$\Delta f_4 =$	0.0625	0.0445	0.0214	0.0084	0.0045 f

Example. It is required to design a suspension bridge with steel-wire cables to carry a roadway 12 meters wide. Span $l = 100$ m $= (328$ ft.), rise $f = 8$ m. Two cables are used. The loads per cable per linear meter are: Dead weight of roadway $= 1400$ kg. ($= 940$ lbs. p. l. f.), live load $= 400 \times 6 = 2400$ kg. ($= 82 \times 19.7 = 1615$ lbs. p. l. f.), hence $q_0 = 3800$ kg.; also the weight of the suspension rods, $j = \frac{q_0}{s} \cdot \frac{\gamma}{10} = \frac{3800}{800} \times \frac{7.8}{10} = 3.7$ kg. $\left(= \frac{q_0}{s} \cdot \frac{\gamma}{144} = \frac{2555}{11{,}390} \times \frac{488}{144} = 7.6 \text{ lbs. per ft. p. l. f.}\right)$ The working stress is taken at $s = 2000$ kg. per sq. cm. ($= 28{,}460$ lbs. per sq. in.). Using $\tan\tau = \frac{4f}{l} = 0.32$, $\sec\tau = 1.050$, the value of the uniform cable cross-section is given by equ. (29) as $A = 333.7$ cm.^{2} ($= 51.7$ sq. in.). The weight of a cable is therefore $g = 0.78 \cdot A = 260.3$ kg. per m. ($= 175$ lbs. per ft.), and the horizontal tension is, by equ. (26), $H = 635.6$ tonnes ($= 1{,}402{,}000$ lbs.) The angle of suspension is now more accurately given by equ. (28) as $\tan\tau = 0.32024$; and the length of the cable is given by equ. (30) as $L = 101.672$ m.

If there is no stiffening construction, the deflections are calculated as follows: The maximum crown deflection under partial (symmetrical) loading is given by equ. (38) $\left(\text{for } \frac{p}{g} = \frac{2400}{1671}\right)$ as

$$\Delta f_1 = 0.057 \cdot f = 0.456 \text{ m.}$$

If the horizontal lengths of the backstays are taken as $l_1 = 25$ m., then the crown deflection due to elastic stretch under full load is given by equs. (44) and (41),

$$\Delta f_2 = 0.208 \text{ m.}$$

The deflection due to a temperature variation of $\pm$ 30° C. is, by equs. (44) and (42),

$$\Delta f_t = \pm 0.144 \text{ m.}$$

Finally, by equ. (47), the largest horizontal displacement of the crown will be $e = 0.110\, l = 11$ m., and the corresponding uplift at the center of the span is, by equ. (48), $\Delta f_4 = 0.0325\, f = 0.260$ m. The total vertical displacement at the center of the span caused by the crossing of the maximum live load amounts therefore, in the unstiffened suspension bridge, approximately to $+0.46$ m. and -0.26 m., i. e., 0.72 m., to which must be added the temperature deflections of $\pm\, 0.144$ m.

5. Secondary Stresses. The preceding theory of the flexible cable is based on the assumption that the cable (or chain) is constantly free to assume the curve of equilibrium corresponding to the momentary loading; this neglects the resistance to deformation offered by the friction in the hinges of the chain or the stiffness of the wire of the cable. Although the latter effect may be so small as to have no appreciable influence on the stresses in the cable, it will generally be otherwise with the frictional resistances in the chain-hinges; experiments made on existing bridges by *Steiner* and *Frankel* with the aid of Frankel's Extensometer, have demonstrated the occurrence of considerable bending strains in the individual links of the chain.

If T denotes the axial tension in one of the links, d the diameter of the pin at the hinge, and ϕ the coefficient of friction, then the bending moment transmitted to the link by friction may reach the value $M = \phi \,.\, T \,.\, \frac{d}{2}$. Approximately we may take $\phi \,.\, T = \phi' \,.\, H$, where, in the most severely stressed link, $\phi' = \phi \left(1 + 8\, \frac{f^2}{l^2}\right)$. Then, to obtain the greatest possible value of the bending moment, H must be determined for that intensity of loading which, applied asymmetrically, will produce moments of deformation equal to the friction-moment $\phi' \,.\, H \,.\, \frac{d}{2}$. For this purpose, we may consider the live load of p per linear unit resolved into two parts p' and p'', where the former covers the span only partially so as to produce the bending moment given by equ. (104ᵃ) as $M = 0.0165\, p'\, l^2$; while the second part p'' is supposed to cover the entire span. Then, neglecting the effect of the load p' upon the value of H, we have

$$0.0165 p' l^2 = \phi' . \frac{1}{8} (p'' + g) \frac{l^2}{f} \cdot \frac{d}{2},$$

whence

$$p'' + g = \frac{0.264}{0.264 + \phi' \dfrac{d}{f}} (p + g).$$

Hence, the maximum bending moment will be

$$M = \frac{\phi' . d}{16} \cdot \frac{0.264}{0.264 + \phi' \dfrac{d}{f}} (p + g) \cdot \frac{l^2}{f}.$$

Thus, knowing the sectional dimensions of the members of the chain, we may readily determine the bending stresses. For the friction coefficient ϕ, particularly in old chain bridges where rust may occur, a high value should be adopted, at least $\phi = 0.20$.

Example. In an existing chain-bridge of 275 ft. span and 18.6 ft. rise there is carried by each chain its own weight of $g = 650$ lbs. p. l. f. together with an applied load of $p = 820$ lbs. p. l. f. For these values $H_{g+p} = 746{,}000$ lbs. The diameter of the pins amounts to $d = 0.164$ ft. Using $\phi' = 0.25$, we find $\dfrac{0.264}{0.264 + \phi' \frac{d}{f}} = 0.99$ and $M = 0.25 \times 0.082 \times 746{,}000 \times 0.99 =$ 15,140 ft. lbs. The chain consists of 5 eye-bars, each 5.2 inches deep and 1.22 inches thick; hence each bar must take a bending moment of $\dfrac{15{,}140 \times 12}{5} = 36{,}330$ in. lbs., and with a section modulus of 5.5 in^3., the resulting maximum fiber stress will be $\dfrac{36{,}330}{5.5} = 6{,}600$ lbs. per sq. in.

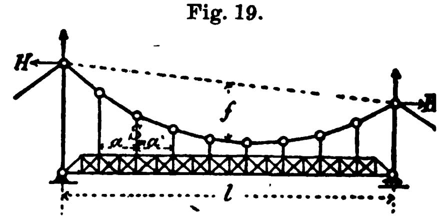
B. The Stiffened Suspension Bridge.

§5. **Approximate Theory.** In order to reduce the static distortions of the funicular polygon or flexible cable discussed in §4, there is introduced a straight truss connected to the cable by suspension rods. This stiffening truss may either extend over a single span, i. e., simply rest on two supports, or it may be built continuous over several spans. In the latter form, the only cases of practical application are those of two equal spans or of three spans. As already mentioned in the Introduction, this type of structure is statically indeterminate; but, by introducing a hinge in the stiffening truss, we may either secure determinateness or at least reduce the degree of indeterminateness.

We first adopt the assumption that the truss is sufficiently stiff to render the deformations of the cable due to moving load practically negligible; in other words, we assume, as in all other rigid structures, that the lever-arms of the applied forces are not altered by the deformations of the system. This *approximate theory* will usually be sufficiently accurate for all practical purposes; but, in order to determine the limits of its applicability, a method for a more exact design will be appended.

1. **Conditions for Equilibrium of the Funicular Polygon.** Equations (1) to (5), (§3), will also apply here if the forces P in those equations are put equal to the stresses of the suspension rods S increased by the panel-loads K corresponding to the dead weight of the cable. If the rods are located at uniform intervals a, then we have by equ. (3^a)

Fig. 19.



$$-H . \Delta^2 y_m = (S_m + K) . a \qquad (49.$$

If the vertices of the funicular polygon lie along a parabola with a vertical axis, and if the adjacent rods are distant a and a' from the m^{th} rod, then equ. (3^b) gives,

$$H . \frac{4f\,(a + a')}{l^2} = S_m + K \qquad (50.$$

Hence, if the cable is parabolic, and if the panel-points are uniformly spaced (horizontally), the suspender-forces must be

uniform throughout. It thus becomes the function of the stiffening-truss to distribute any arbitrary live load in whatever manner may be demanded by the particular form of the cable-curve.

If the cable is continuously curved, then the applied loads per horizontal linear unit are given by

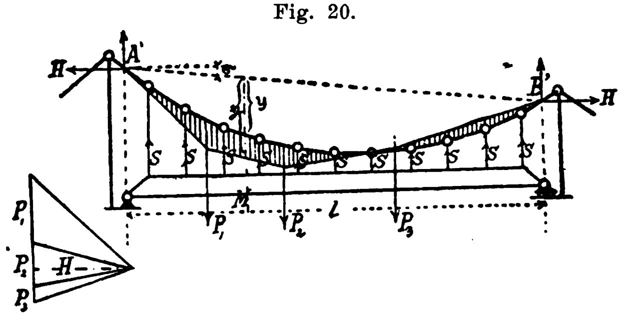
$$s_m + k = -H.\frac{d^2y}{dx^2} \quad \text{(51.}$$

and if the curve is a parabola, the uniform loading becomes

$$s + k = H.\frac{8f}{l^2} \quad \text{(52.}$$

The approximate theory assumes that the above equations (49.) to (52) hold true even after deformation. This assumption is all the more admissible the stiffer the truss, for the less will then be the deflections transmitted by the rods to the cable.

Fig. 20.



2. Forces Acting on the Stiffening Truss. These comprise the dead weight of the truss and any construction it may carry, the live load, and the stresses in the suspenders which are, in effect, vertical loads directed upward. If we imagine the last forces removed, then the bending moment **M** and the shear **S** at any section of the truss distant x from the left end may be determined exactly as for an ordinary beam (simple or continuous according as the truss rests on two or more supports). This moment and shear would be produced if the cable did not exist and the entire load were carried by the truss alone. If $-M_s$ represents the bending moment of the suspender-forces at the section considered, then the total moment in the stiffening truss will be $M = \mathbf{M} - M_s$.

Let the straight line passing through A' and B', the points of the cable directly above the ends of the truss (Fig. 20), be adopted as a coordinate axis from which to measure the ordinates y of the funicular polygon. Also consider the dead weight

of the cable (k p. l. f.) to be uniformly distributed. We then have the following cases:

α.) For a truss simply resting on two supports, with span $A\,B = l$, by a familiar property of the funicular polygon,

$$M_s + \frac{1}{2} kx\,(l-x) = H\,.\,y;$$

consequently

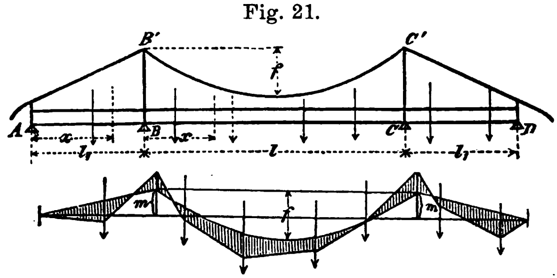
$$M = \mathbf{M} + \frac{1}{2}\,k\,x\,(l-x) - H\,.\,y \ldots\ldots\ldots\ldots (53.$$

Considering the cable-weight as included in the load carried by the stiffening truss, also representing **M** by the ordinates **y** of an equilibrium polygon or curve constructed for the applied loading with a pole distance $= H$, we have

$$M = \mathbf{M} - H\,.\,y = H\,.\,(\mathbf{y} - y) \ldots\ldots\ldots\ldots (54.$$

Hence the bending moment at any section of the stiffening truss is proportional to the vertical intercept between the axis

Fig. 21.



of the cable and the equilibrium polygon for the applied loads drawn through the points A' B' (Fig. 20).

β.) For a stiffening truss continuous over several spans,

$$M_s = H\,.\,y - M''_s \frac{x}{l} - M'_s \frac{l-x}{l} = H\left(y - m''\frac{x}{l} - m'\frac{l-x}{l}\right),$$

hence

$$M = \mathbf{M} - H\left(y - m''\,\frac{x}{l} - m'\,\frac{l-x}{l}\right). \ldots\ldots\ldots\ldots (55.$$

Here M'_s and M''_s denote the end bending moments produced by the suspender forces acting on the continuous truss, introduced with their appropriate negative signs. These moments may readily be represented by the lengths m' and m'', with a pole distance $H = 1$, for any given form of cable.

In the case represented in Fig. 21, where the truss extends

symmetrically over two side spans but without connection to the backstays, we have

$$\left.\begin{array}{l}\text{in the main span, } M = \mathbf{M} - H\,(y - m) \\ \text{in the side span, } M = \mathbf{M} + H\,.\,m\,.\,\frac{x}{l_1}\end{array}\right\}\ldots (56.$$

With a uniform distribution of the suspender forces, i. e., for a parabolic cable-curve or chain-polygon, if the ratio of span-lengths is $r = \frac{l_1}{l}$ and if $i = \frac{I}{I_1}$ is the ratio between the moments of inertia of the cross-sections of the stiffening truss in the main and side spans respectively, the theorem of three moments gives

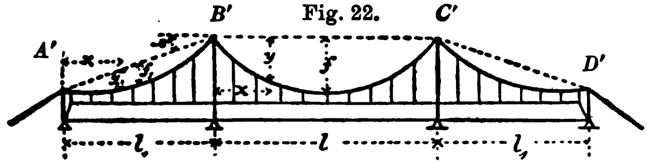
$$m = \frac{2}{3 + 2\,ir}\,.\,f \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (57.$$

Denoting the coefficient of f in the above equation by the symbol ϵ (a constant for any given structure), $m = \epsilon\,.\,f$, hence

$$\left.\begin{array}{l}\text{in the main span, } M = \mathbf{M} - H\,(y - \epsilon\,.\,f) \\ \text{in the side span, } M = \mathbf{M} + H\,.\,\frac{x}{l_1}\,.\,\epsilon f\end{array}\right\}\ldots (57^a.$$

If the truss is connected to the cable by *suspension rods in the side spans also* (Fig. 22), then, with a parabolic cable curve,

Fig. 22.



$$m = \frac{2\,(1 + ir^3)}{3 + 2\,ir}\,.\,f \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (58.$$

As before, let us designate the coefficient of f in this equation by the constant ϵ, so that $m = \epsilon\,.\,f$. We then obtain,

$$\left.\begin{array}{l}\text{for the main span, } M = \mathbf{M} - H\,(y - \epsilon\,.\,f) \\ \text{for the side spans, } M = \mathbf{M} - H\,(y_1 - \frac{x}{l_1}\,.\,\epsilon\,.\,f)\end{array}\right\}\ldots (58^a.$$

where y_1 represents the ordinates of the side-cable below the connecting chord $A'\,B'$.

The transverse shears S are given by the following equations:

a.) In the single-span stiffening truss

$$S = \mathbf{S} - H\,(tan\,\tau - tan\,\sigma) \ldots\ldots\ldots\ldots\ldots\ldots (59.$$

β.) In the continuous stiffening truss,

$$S = \mathbf{S} - H\left(tan\ \tau - tan\ \sigma + \frac{m'-m''}{l}\right) \ldots\ldots\ldots\ldots (60.$$

Here τ denotes the inclination to the horizontal of the tangent to the cable curve at the given section, and σ is the inclination to the horizontal of the cable-chord joining the points of suspension; both of these angles are reckoned positive when directed downwards to the right. m' and m'' have the significance previously assigned. For the case represented by Fig. 21, the shears will be

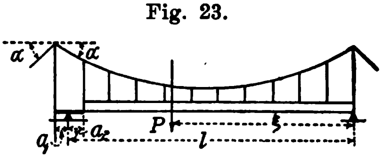
$$\left.\begin{array}{l} \text{in the main span, } S = \mathbf{S} - H\ tan\ \tau \\ \text{in the side spans, } S = \mathbf{S} + H \cdot \dfrac{2}{3+2\,ir} \cdot \dfrac{f}{l_1} \end{array}\right\} \ldots (61.$$

If the truss is suspended in the side spans also, the second of equs. (61) is replaced by

$$S = \mathbf{S} + H\left(\frac{2+2\,ir^2}{3+2\,ir} \cdot \frac{f}{l_1} + tan\ \sigma - tan\ \tau\right) \ldots (62.$$

3. Determination of the Horizontal Tension H. In the bridge systems considered above, the horizontal tension is statically indeterminate unless another condition is furnished by some special construction so as to eliminate the indeterminateness in this respect. For instance, the cable might be given a definite, invariable H by replacing one of the anchorages by an attached weight. The stiffening truss would then be subjected to an unchanging upward loading.—Another arrangement that has been proposed, but never yet executed, is to balance the end-reactions or to fix the ratio of the cable and truss reactions by supporting both of these members on the ends of a common lever. If $n = \frac{a_1}{a_2}$ is the ratio of the lever arm of the truss to that of the cable-tower, then, with the arrangement and loading shown in Fig. 23,

Fig. 23.



$$H = \frac{P \cdot \xi}{l} \cdot \frac{n}{2+n}\ cotan\ \alpha \ldots\ldots\ldots\ldots\ldots\ldots (63.$$

In actual practice, however, the only cases of importance are the following two arrangements:

a.) *The stiffening truss is provided with a central hinge.* This furnishes a condition which enables H to be directly determined; viz., at the section through the hinge the moment M must equal zero. Consequently, if the bending moment at the same section of a simple beam is denoted by $\mathbf{M}_0$, and if f is the

ordinate of the corresponding point of the cable, then for a single-span bridge, by equ. (54)

$$H = \frac{M_0}{f} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (64.$$

and for a *continuous* truss having a hinge in the central span, by equ. (56)

$$H = \frac{M_0}{f - m} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (65.$$

b.) *The stiffening truss has no central hinge,* and is not provided with any of the devices, described above, for making it statically determinate. The required equation for the determination of the horizontal tension must therefore be deduced from the elastic deformations of the system. Various procedures may be adopted for this purpose. We may, for example, equate the variations in the cable-ordinates, increased by the elongations of the suspension rods, to the deflections of the stiffening truss, and thus develop an expression for H. A simpler method, however, consists in applying the "Theorem of Least Work."

If N denotes the axial stress and M the bending moment at any section of an individual member of the system, and if s is the length of the member, A its cross-section, I its moment of inertia about the neutral axis, and E its coefficient of elasticity, then the familiar expression for the work of deformation is

$$W = \int \frac{N^2}{2\,EA}\, ds + \int \frac{M^2}{2\,EI}\, ds = min \ldots\ldots\ldots\ldots (66.$$

and the resulting equation of condition for the case under consideration is

$$\frac{dW}{dH} = \int \frac{N}{EA} \cdot \frac{dN}{dH} \cdot ds + \int \frac{M}{EI} \cdot \frac{dM}{dH} \cdot ds = 0 \ldots\ldots (67.$$

To apply this equation, it is simply necessary to express the axial stress and bending moment for each part of the system in terms of the external forces and the unknown horizontal tension H.

a.) Let us first consider the type of construction shown in Fig. 19, consisting of a suspension bridge with stiffening truss extending over but one span. Let

A = cross-section of the cable at any point.
A_0 = cross-section of the cable at the crown.
λ = length of cable between consecutive rods.
a = horizontal distance between consecutive rods.
$2\,A_1$ = cross-section of the stiffening truss.
(= cross-section of the two chords.)
I = moment of inertia of stiffening truss.

A_2 = cross-section of a suspension rod.
y' = length of a suspension rod.
y = ordinate to cable measured below the closing chord.
h = effective depth of the stiffening truss.
(= distance between the neutral axes of the two chords.)
f = the versine (or rise) of the cable.
f' = height of the end posts or towers.
A_3 = cross-section of end posts or towers.
l_1 = horizontal projection of the backstays.
a_1 = inclination to horizontal of backstays.
a = angle of suspension of the cable.
G = total load on the stiffening truss.
A, B = resulting vertical end reactions.
K = weight of cable per panel point.
k = weight of cable per linear foot.

Remembering that the stress in the cable (or chain) is

$$T = H \cdot \frac{\lambda}{a} \dots\dots\dots\dots\dots\dots\dots\dots \quad (68.$$

and assuming that the cross-section at every point is proportional to the stress, then we have $A = A_0 \cdot \frac{\lambda}{a}$ and for the backstays $A' = A_0 . sec\, a_1$.

We must now write out the expression of equ. (67) for each member of the system, for which purpose the following tabulation will be used. In this table, the stress in the suspension rods is taken from equ. (49); and it is assumed either

a.) that the stiffening truss is not independently supported but is hung, at the ends, from the points of suspension of the cable, or

b.) that the stiffening truss has independent supports, one fixed and the other movable horizontally, which, on account of the possible negative reactions, must be considered as anchored down.

Member	N or M	$\frac{dN}{dH}$ or $\frac{dM}{dH}$	A or I	$\int ds$	$E\frac{dW}{dH}$
Cable........	$H \cdot \frac{\lambda}{a}$	$\frac{\lambda}{a}$	$\frac{\lambda}{a} \cdot A_0$	λ	$\frac{H}{aA_0} \cdot \sum_0^l \lambda^2$
Backstays....	$H \cdot \sec a_1$	$\sec a_1$	$A_0 \cdot \sec a_1$	$2l_1 \cdot \sec a_1$	$\frac{2H}{A_0} \cdot l_1 \sec^2 a_1$
Suspension Rods.......	$H\frac{\Delta^2 y}{a} - K$	$\frac{\Delta^2 y}{a}$	A_2	y'	$\frac{H}{a^2 A_2}\sum_0^l y'(\Delta^2 y)^2 - \frac{K}{aA_2}\sum_0^l y' \Delta^2 y$
a) End Posts..	$\begin{cases} A - H \tan a \\ B - H \tan a \end{cases}$	$-\tan a$	A_3	$2f'$	$\frac{2H}{A_3} f' \tan^2 a - \frac{G}{A_3} \cdot f' \tan a$
b) Towers....	$H(\tan a + \tan a_1)$	$(\tan a + \tan a_1)$			$\frac{2H}{A_3} \cdot f'(\tan a + \tan a_1)^2 \cdot \frac{E}{E_1}$
Stiffening Truss......	$\mathbf{M} + \frac{1}{2}kx \cdot (l-x) - Hy$	$-y$	I	$\int dx$	$H\int_0^l \frac{y^2 dx}{I} - \int_0^l \frac{\mathbf{M}y}{I}dx - \frac{1}{2}k\int_0^l \frac{yx(l-x)}{I}dx$

Noting that $\sum_0^l \lambda^2 = \sum_0^l a^2 + \sum_0^l (\Delta y)^2 = l . a - \sum_0^l y \, \Delta^2 y$, and that for an approximately parabolic curve of cable we may write $\int \frac{yx\,(l-x)}{I} \cdot dx = \frac{l^2}{4f} \int \frac{y^2\,dx}{I}$, we obtain from the relation $\frac{dW}{dH} = 0$ the following expressions for the horizontal tension:

For the case a):

$$H = \frac{A_0 \int_0^l \frac{\mathrm{M}y}{I} dx + f' \tan \alpha \frac{A_0}{A_3} G + \left(\frac{A_0}{A_2} \sum_0^l y' . \Delta^2 y + \frac{A_0 l^2}{8f} \int_0^l \frac{y^2\,dx}{I} \right) k}{l - \sum_0^l \frac{y}{a} \Delta^2 y + 2 l_1 \sec^2 \alpha_1 + \frac{A_0}{A_2} \sum_0^l \frac{y'}{a^2} (\Delta^2 y)^2 + 2 f' \frac{A_0}{A_3} \tan^2 \alpha + A_0 \int_0^l \frac{y^2\,dx}{I}} \quad \ldots\ldots\ldots\ldots (69.$$

For the case b):

$$H = \frac{A_0 \int_0^l \frac{\mathrm{M}y}{I}\, dx + \left(\frac{A_0}{A_2} \sum_0^l y' . \Delta^2 y + \frac{A_0 l^2}{8f} \int_0^l \frac{y^2\,dx}{I} \right) k}{l - \sum_0^l \frac{y}{a} \Delta^2 y + 2 l_1 \sec^2 \alpha_1 + \frac{A_0}{A_2} \sum_0^l \frac{y'}{a^2} (\Delta^2 y)^2 + 2f' \frac{A_0}{A_3} \frac{E}{E_1} \left(\tan \alpha + \tan \alpha_1 \right)^2 + A_0 \int_0^l \frac{y^2\,dx}{I}} \quad \ldots\ldots\ldots\ldots (70.$$

Here E_1 is the elastic coefficient for compression of the towers.

Fig. 24.

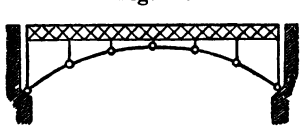
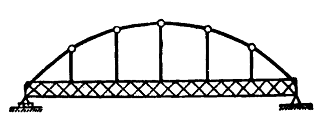


Fig. 25.



Putting $l_1 = 0$, equ. (69) gives also the horizontal thrust for the arched-polygon with stiffening truss (Fig. 24).

Equ. (69) may also be applied to the form of structure represented in Fig. 25, by putting $l_1 = 0$, $f' = 0$, $y' = y$, and adding to the denominator of the expression the term $\frac{A_0}{2A_1} \cdot l$, representing the axial tension in the stiffening truss.

The same term must also be appended to the denominators of equs. (69) and (70), at the same time putting $l_1 = 0$, if instead of the backstays there is introduced a strut of section $2A_1$ connecting the points of suspension of the cable.

Dividing numerator and denominator of the preceding equations by A_0, and collecting the terms that do not contain I under the symbols Q and q respectively, we obtain the following general expression for H:

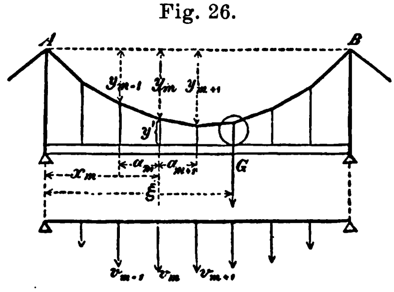
$$H = \frac{\int_0^l \frac{\mathrm{M}\, y}{I} dx + \frac{Q}{A_0} + \frac{kl^2}{8f} \int_0^l \frac{y^2\, dx}{I}}{\int_0^l \frac{y^2\, dx}{I} + \frac{q}{A_0}} \qquad \text{(71.}$$

As the second term of the numerator is usually negligible compared to the first term, we have very nearly

$$H = \frac{\int_0^l \frac{\mathrm{M}\, y}{I} dx}{\int_0^l \frac{y^2\, dx}{I} + \frac{q}{A_0}} + \frac{kl^2}{8f}\left(1 - n\right) \qquad \text{(71}^{\text{a}}\text{.}$$

where n denotes the very small quantity $\frac{q}{A_0 \int_0^l \frac{y^2\, dx}{I}}$.

Fig. 26.



We now introduce the assumption that the moment of inertia of the truss, I, is constant within any panel-length a, and

its value in any panel between the $(m-1)$th and (m)th suspension rods will be denoted by I_m. Then the definite integrals in the above formula may be resolved into summations, the terms for which are obtained by extending the integration to each separate panel. In this process we may use the values

$$y = y_{m-1} + \frac{x'}{a_m}(y_m - y_{m-1}),$$

and

$$\mathbf{M} = \mathbf{M}_{m-1} + \frac{x'}{a_m}(\mathbf{M}_m - \mathbf{M}_{m-1}),$$

where $\mathbf{M}_{m-1}$ and $\mathbf{M}_m$ are the ordinary beam bending moments at the panel-points $m-1$ and m, and the abscissae x' are measured from the $(m-1)$th panel point. We then obtain

$$\frac{1}{I_m}\int_0^{a_m} \mathbf{M} y\, dx = \frac{a_m}{6 I_m}\left[\mathbf{M}_{m-1}(2y_{m-1} + y_m) + \mathbf{M}_m(2y_m + y_{m-1})\right]$$

and

$$\frac{1}{I_m}\int_0^{a_m} y^2\, dx = \frac{a_m}{6 I_m}\left[y_{m-1}(2y_{m-1} + y_m) + y_m(2y_m + y_{m-1})\right].$$

Adopting a mean moment of inertia I_0 and a mean panel-length a_0, we obtain the following summation expressions:

$$\int_0^l \frac{\mathbf{M} y\, dx}{I} = \frac{a_0}{6 I_0}\Big[\ldots + \frac{a_m}{a_0}\cdot\frac{I_0}{I_m}\Big\{\mathbf{M}_{m-1}(2y_{m-1} + y_m)$$

$$+ \mathbf{M}_m(2y_m + y_{m-1})\Big\} + \frac{a_{m+1}}{a_0}\frac{I_0}{I_{m+1}}\Big\{\mathbf{M}_m(2y_m + y_{m+1})$$

$$+ \mathbf{M}_{m+1}(2y_{m+1} + y_m)\Big\} + \ldots\Big] = \frac{a_0}{6 I_0}\sum_0^l \mathbf{M}_m$$

$$\cdot\left[\frac{a_m}{a_0}\frac{I_0}{I_m}(2y_m + y_{m-1}) + \frac{a_{m+1}}{a_0}\frac{I_0}{I_{m+1}}(2y_m + y_{m+1})\right].$$

and, similarly,

$$\int_0^l \frac{y^2\, dx}{I} = \frac{a_0}{6 I_0}\sum_0^l y_m\Big[\frac{a_m}{a_0}\frac{I_0}{I_m}(2y_m + y_{m-1})$$

$$+ \frac{a_{m+1}}{a_0}\frac{I_0}{I_{m+1}}(2y_m + y_{m+1})\Big].$$

Introducing, for abbreviation, the symbol

$$\left.\begin{aligned} v_m = & \frac{a_m}{6\,a_0}\,\frac{I_0}{I_m}\,(2\,y_m + y_{m-1}) \\ & + \frac{a_{m+1}}{6\,a_0}\,\frac{I_0}{I_{m+1}}\,(2\,y_m + y_{m+1}), \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (72.$$

and substituting in equ. (71^a), we obtain a final expression for the horizontal tension due to the external loading:

$$H = \frac{\sum\limits_0^l \mathbf{M}_m\, v_m}{\sum\limits_0^l y_m\, v_m + \frac{q}{a_0}\,\frac{I_0}{A_0}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (73.$$

If the loading consists merely of a concentration G, distant ξ from the left support, then the two summations in the above expression may be given a static significance, enabling them to be determined graphically. Thus, if x_m is the abscissa of any panel point,

$$\sum_0^l \mathbf{M}_m\, v_m = G\left[\frac{l-\xi}{l}\sum_0^\xi x_m\, v_m + \frac{\xi}{l}\sum_\xi^l\,(l - x_m)\,v_m\right] = G\,.\,m_\xi$$

$$\text{and} \quad \sum_0^l y_m\, v_m = \mu,$$

where m_ξ signifies the moment, at the load-point (ξ), producible in a simply supported beam loaded at every panel point (m) with a vertical force of magnitude v_m. Similarly, μ is the static moment, about the chord $A\,B$, of the forces v conceived as acting horizontally at the respective panel points. These moments may readily be constructed as the ordinates of the proper funicular polygons.

If the panel-length a is constant, then

$$v_m = \frac{I_0}{6\,I_m}\,(2\,y_m + y_{m-1}) + \frac{I_0}{6\,I_{m+1}}\,(2\,y_m + y_{m+1}),$$

and if the moment of inertia, I, is also constant,

$$v_m = \frac{1}{6}\,(y_{m-1} + 4\,y_m + y_{m+1}),$$

or, with sufficient accuracy,

$$v_m = y_m.$$

The graphic construction is executed exactly as in Fig. 54, (see §17). We first divide the area between the cable and its chord into strips of width a, which may be taken equal to the spacing of the suspension rods; and the load-elements v are

either computed by equ. (72) or assumed approximately equal to the ordinates y. With the pole distance p, we next construct the two funicular polygons b and c for the forces v acting vertically and horizontally, respectively. The ordinate of the polygon b, directly under the point of application of the load, gives the value of H, if the concentration G is represented by a length composed of the intercept $n_0 n$ of the polygon c on the chord $A\,B$ plus the length $nn_1 = c = \frac{g}{a_0 p} \frac{I_0}{A_0}$. Thus,

$$H = \frac{m_\xi}{\mu + c\,p} \cdot G \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (74.$$

In an actual design, the data were: $l = 50$ m., $f = 6.5$ m., $\Sigma \frac{y}{a} \Delta^2 y = 4.506$ m., $l_1 = 15$m., $\frac{I_0}{A_0} = 2.049$; the effect of the suspension rods was neglected. Choosing $a = 2.5$ m., $p = 35$ m., there was obtained $c = 2.167$ m. For a load at the center of the span, the construction yielded $H = \frac{19.3}{15.0} = 1.286\,G$.

If the stiffening beam is a truss with parallel chords, then in equs. (71) to (74) we may write $I = A_1 \cdot \frac{h^2}{2}$. Again I may be assumed constant either throughout each panel a or, for a first approximation, throughout the entire span. If we desire, furthermore, to include the effect of the web members, if A_4 is the section of any web member and γ its inclination to the vertical, then, by equ. (59), we must correct equs. (69) and (70) by adding to the numerators of the expressions for H the term, $+ \frac{h}{a} sec^3\gamma \cdot \sum_0^l S \frac{A_0}{A_4} . \Delta y$, and to the denominators the term, $+ \frac{h}{a^2} sec^3 \gamma \cdot \sum_0^l \frac{A_0}{A_4} (\Delta y)^2$. The following values, therefore, must be substituted in equ. (71) and the subsequent formulae, for the case (b):

$$Q = + k \frac{A_0}{A_2} . \sum_0^l y' \, \Delta^2 y + \frac{h}{a} sec^3 \gamma \sum_0^l S \frac{A_0}{A_4} . \Delta y \,,$$

$$q = l - \sum_0^l \frac{y}{a} \Delta^2 y + 2\,l_1 \, sec^2 \alpha_1 + \frac{A_0}{A_2} \Sigma \frac{y'}{a^2} (\Delta^2 y)^2$$

$$+ 2 f' \frac{A_0}{A_3} . \frac{E}{E_1} (tan\, \alpha + tan\, \alpha_1)^2 + \frac{h}{a^2} sec^3 \gamma \sum_0^l \frac{A_0}{A_4} . (\Delta y)^2.$$

However, in the total work of deformation, the influence of the web members of the stiffening truss, as well as that of the suspension rods, is so slight that the corresponding terms may,

without appreciable error, be neglected. Under the usual conditions of practice, their effect on the value of H cannot exceed 0.3 to 0.5%.

If the cable or arch is parabolic in form, then, taking the origin of coordinates at the left point of support, the equation of the curve will be

$$y = \frac{4f}{l^2}(l-x) \, . \, x.$$

If, furthermore, the distances a between suspension rods are assumed vanishingly small, the summations in the expression for H may be replaced by the corresponding integrations, thus yielding the following values:

$$\sum_0^l \frac{\lambda^2}{a} = \int_0^l \left(\frac{dx^2+dy^2}{dx^2}\right) \cdot dx = l\left(1+\frac{16}{3}\frac{f^2}{l^2}\right)$$

$$\sum_0^l \frac{y'}{a^3}(\Delta^2 y)^2 = \int_0^l (f'-y)\left(\frac{d^2 y}{dx^2}\right)^2 . \, dx = \frac{64 f^2}{l^3}\left(f'-\frac{2}{3}f\right)$$

$$\int_0^l y^2 \, dx = \frac{8}{15} f^2 \cdot l$$

$$\sum_0^l \frac{y'}{a}\Delta^2 y = \int_0^l (f'-y)\frac{d^2 y}{dx^2} \cdot dx = \frac{8f}{l}\left(f'-\frac{2}{3}f\right)$$

$$\sum_0^l \frac{y}{a}\Delta^2 y = \int_0^l y \cdot \frac{d^2 y}{dx^2} . \, dx = \frac{16}{3}\frac{f^2}{l} \, .$$

Finally, if the moment of inertia of the stiffening truss is assumed constant ($=I$) for the entire span, and if the loading consists of a concentration G located at a distance ξ from the left end of the span, then we have

$$\int_0^l \mathrm{M} . y . dx = G \cdot \frac{4f}{l^2}\left[(l-\xi) \cdot \int_0^\xi x^2(l-x)\,dx + \xi \cdot \int_\xi^l x(l-x)^2 . dx\right]$$

$$= \frac{1}{3}\xi\,(\xi^3 - 2l\,\xi^2 + l^3)\,\frac{f}{l^2} \cdot G.$$

Substituting the above values for the summations of equ. (70), we obtain the following expression for the horizontal tension:

$$H = \frac{l}{f} \cdot \frac{\frac{\xi}{l}\left[\left(\frac{\xi}{l}\right)^3 - 2\left(\frac{\xi}{l}\right)^2 + 1\right]G - \left[8\,(3f'-2f)\frac{I}{\mathbf{A}_2 l^4} - \frac{1}{5}\right]kl}{\frac{8}{5} + 3\frac{I}{A_0 f^2}\left(1 + \frac{16}{3}\frac{f^2}{l^2} + 2 - sec^2 \alpha_1\right) + 64\,(3f'-2f)\frac{I}{\mathbf{A}_2 l^4} + \frac{6\,I\,f'}{A_3\,l^3}\frac{E}{E}\left(4 + \frac{l}{f} tan\ \alpha_1\right)} \quad (75.$$

In this expression, $\mathbf{A}_2$ denotes the sectional area of the suspension rods per linear foot of span, i. e., $\frac{A_2}{a}$.

Neglecting the effect of the elongation of the rods, as well as that of the compression of the towers, and omitting the cable-weight from consideration, we have

$$H = \frac{\frac{\xi}{l}\left(\frac{\xi^3}{l^3} - 2\frac{\xi^2}{l^2} + 1\right) . G\frac{l}{f}}{\frac{8}{5} + 3\frac{I}{A_0 . f^2}\left(1 + \frac{16}{3}\frac{f^2}{l^2} + 2\frac{l_1}{l} sec^2 \alpha_1\right)} \ldots\ldots \quad (75^a.$$

The last equation will usually be accurate enough for a preliminary design; on the basis of the results thus obtained, an estimate can be made of the probable variation of the moments of inertia (I), and a re-design may then be executed by the more exact equation (71) or (73). Of course it will be necessary, in the first design, to guess the probable value of the ratio I/A_0, but the range of this quantity in practice lies between such narrow limits that the corresponding variation in H is quite inappreciable. If we denote the ratio $\frac{I}{A_0 f^2}$ by the symbol n, and if Δn is its variation, then approximately $\frac{\Delta H}{H} = -\frac{30}{8+30n} \cdot \Delta n$. As may be shown by a study of actual designs, the ratio n always lies between the limits 0.01 and 0.02, so that, at the mean value, $\frac{\Delta H}{H} = 3.55 \ . \Delta n$; consequently, if the variation of n is 0.01, the resulting error in H will not be more than 3.6%.

If the horizontal span of the cable l' differs from the span of the truss l, we must substitute l' for l in the first two terms of the denominator in equ. (69) or (70), also multiply the terms containing $\frac{I}{A_0}$ in the denominator of equ. (75) by $\frac{l'}{l}$. In this case, f is the versine of the cable for the span l.

β.) By the same analysis as was applied in deriving equ. (69), we may also obtain the expression for H for a three-span suspension bridge (Fig. 27). It will here suffice to derive the approximate equations based on the assumptions of a parabolic

weightless cable, and a stiffening truss with moment of inertia constant throughout each span.

In the following, let

f = the versine of the cable in the main span (l),

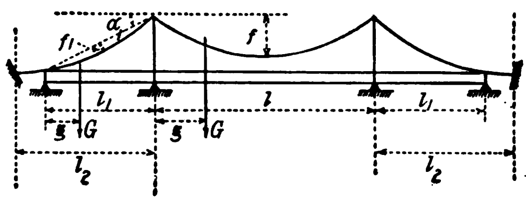
f_1 = the versine of the cable in the side spans (l_1), measured vertically,*

l_2 = the horizontal distance from tower to anchorage,

α_1 = inclination to the horizontal of the cable-chord in the side span,

$\left.\begin{matrix} I \\ I_1 \end{matrix}\right\}$ = moments of inertia of the stiffening truss in main and side spans respectively.

Fig. 27.



Then, for a concentration G in the main span, distant ξ from either tower,

$$H = \left.\frac{\left(\frac{\xi}{l}\right)^3 - 2\left(\frac{\xi}{l}\right)^2 + 1 - \frac{3}{2}\varepsilon\left(1-\frac{\xi}{l}\right) - \frac{3lI_1}{3lI_1+2l_1I}\left(1-\frac{\xi}{l}\right)\left[1-\frac{3}{2}\varepsilon+\left(\frac{f_1}{f}-\varepsilon\right)\frac{l_1}{l}\cdot\frac{I}{I_1}\right]}{\frac{8}{5}-4\varepsilon+3\varepsilon^2+2\frac{I}{I_1}\frac{l_1}{l}\left(\frac{8}{5}\frac{f_1^2}{f^2}-2\varepsilon\frac{f_1}{f}+\varepsilon^2\right)+\frac{3I}{A_0f^2}\left[1+\frac{16}{3}\frac{f^2}{l^2}+2\frac{l^2}{l}\left(1+\frac{16f_1^2}{3l_1^2}+\tan^2\alpha_1\right)\right]}\cdot\frac{\xi}{f}\cdot G\right\} \quad (76.$$

If the concentration is applied in the side span at a distance ξ from the end-support, the horizontal tension in the cable will be:

$$H = \left.\frac{\left\{\left[\left(\frac{\xi}{l_1}\right)^8 - 2\left(\frac{\xi}{l_1}\right)^2+1\right]\frac{f_1}{f}-\frac{1}{2}\varepsilon\left(1-\left(\frac{\xi}{l_1}\right)^2\right) - \frac{lI_1}{3lI_1+2l_1l}\left(1-\left(\frac{\xi}{l_1}\right)^2\right)\left[1-\frac{3}{2}\varepsilon+\left(\frac{f_1}{f}-\varepsilon\right)\frac{Il_1}{I_1l}\right]\frac{I_1}{I}\right\}\frac{Il_1}{I_1l}}{\frac{8}{5}-4\varepsilon+3\varepsilon^2+2\frac{Il_1}{I_1l}\left(\frac{8}{5}\frac{f_1^2}{f^2}-2\varepsilon\frac{f_1}{f}+\varepsilon^2\right)+\frac{3I}{A_0f^2}\left[1+\frac{16}{3}\frac{f^2}{l^2}+2\frac{l_2}{l}\left(1+\frac{16}{3}\frac{f_1^2}{l_1^2}+\tan^2\alpha_1\right)\right]}\cdot\frac{\xi}{f}G\right\} \ldots (76^a.$$

If the stiffening truss is suspended from the cable in the side spans also, then, in the above equations:

*In Fig. 27, f_1 should be measured vertically, not obliquely as wrongly indicated by the drawing.

$$\varepsilon = \frac{2\left(l^3 + \frac{I}{I_1} l_1^3\right)}{l^2\left(3l + 2\frac{I}{I_1} l_1\right)} \ldots\ldots\ldots\ldots\ldots\ldots \quad (77.$$

But if the stiffening truss, although continuous, is not connected with the cable in the side spans, then, in the above equations, we must put $f_1 = 0$ and

$$\varepsilon = \frac{2l}{3l + 2l_1\frac{I}{I_1}} \ldots\ldots\ldots\ldots\ldots\ldots \quad (78.$$

If, in the latter case, it is desired to include the effect of the elongation of the rods in the main span and the compression of the towers (of height f'), then there must be added to the denominators of the above expressions for H the terms:

$$\frac{3I}{f^2 l}\left[\frac{2E}{E_1 A_1}\left(4\frac{f}{l} + \tan\alpha_1\right)^2 f' + \frac{64}{A_2 l^3}\left(f' - \frac{2}{3}f\right)f^2\right]$$

The last term in the numerators of the above expressions, i. e., the term containing the factor $\frac{l I_1}{3 l I_1 + 2 l_1 I}$, vanishes when the cable curves in the main and side spans are parts of like parabolas, when there are no suspenders in the side spans, or when the stiffening truss is interrupted (i. e., hinged) at the towers. In the last case, viz., the two-hinged suspension bridge, we must also put the factor of continuity $\epsilon = 0$, thus obtaining the following simplified equations:

For a load G in the main span,

$$H = \left.\frac{\left(\frac{\xi}{l}\right)^3 - 2\left(\frac{\xi}{l}\right)^2 + 1}{\frac{8}{5}\left(1 + 2\frac{I l_1}{I_1 l}\cdot\frac{f_1^2}{f^2}\right) + \frac{3I}{A_0 f^2}\left[1 + \frac{16}{3}\frac{f^2}{l^2} + 2\frac{l_1}{l}\left(1 + \frac{16}{3}\frac{f_1^2}{l_1^2} + \tan^2\alpha_1\right)\right]} \cdot \frac{\xi}{f}\cdot G\right\} \ldots \quad (79.$$

For a load G in the side span,

$$H = \left.\frac{\left[\left(\frac{\xi}{l_1}\right)^3 - 2\left(\frac{\xi}{l_1}\right)^2 + 1\right]\frac{I l_1}{I_1 l}\frac{f_1}{f}}{\frac{8}{5}\left(1 + 2\frac{I l_1}{I_1 l}\frac{f_1^2}{f^2}\right) + \frac{3I}{A_0 f^2}\left[1 + \frac{16}{3}\frac{f^2}{l^2} + 2\frac{l_1}{l}\left(1 + \frac{16}{3}\frac{f_1^2}{l_1^2} + \tan^2\alpha_1\right)\right]} \cdot \frac{\xi}{f}\cdot G\right\} . \quad (79_1.$$

To investigate the effect of a moving load, it is advisable to apply the Method of Influence Lines.

On account of their frequent application in the present and succeeding parts of this book, the *general properties of influence lines* will here be reviewed. To obtain the influence line for any given function for which we wish to investigate the law of loading, we erect at each position of the load an ordinate equal to the corresponding value of the function. The resulting curve then indicates definitely the positions at which the influence of the load is maximum or minimum, and also assists in determining the most unfavorable position for a train load. If the influence line is constructed for a moving load equal to unity, then the function in the case of a series of concentrations will be equal to the sum of the products of the loads by the ordinates (Y) of the influence line corresponding to their points of application. If, instead of a unit load, any force G be taken as the basis of the influence line, then for the concentrations P we must take the sum $\frac{1}{G} \Sigma P Y$. The influence line is a continuous curve if the loads are applied directly to the main truss (or other structure). If, on the contrary, the loading may act only on isolated points of the truss, as in the case of principal trusses supporting transverse floor-beams, then the curve is replaced by an inscribed polygon of straight lines with vertices located in the vertical lines through the points of direct loading. (Proof: Let P be the load in any panel a between two transverse beams at which the ordinates for a unit-load influence line are $Y_{\prime}$ and $Y_{\prime\prime}$; let e be the distance of the load from the first of these panel points; then the ordinate Y of the influence line at the point of application of the load must satisfy the equation

$$P \cdot Y = Y_{\prime} \frac{P(a-e)}{a} + Y_{\prime\prime} \frac{P \cdot e}{a},$$

showing that Y is the ordinate of the straight line joining $Y_{\prime}$ and $Y_{\prime\prime}$).

If a train of concentrations is placed arbitrarily upon the structure, it will in general be necessary to shift the entire system of loads one way or the other before they will yield a maximum value of the given function. In order to establish a criterion for the proper direction of this shift or displacement, let $\alpha_1, \alpha_2, \alpha_3, \ldots$, be the angles of inclination of the sides of the influence polygon and let P_1, P_2, P_3 ... be the loads lying within the projections of the respective sides. Then the combined influence of these loads is represented by the sum $P_1 Y_1 + P_2 Y_2 + P_3 Y_3 +$ etc.; and the variation of this function producible by a displacement of the entire train through the distance $\Delta\xi$ (it being assumed that during this displacement no load enters or leaves the span or crosses a panel point) will be given by

$$\Delta\xi\ (P_1 \tan \alpha_1 + P_2 \tan \alpha_2 + P_3 \tan \alpha_3 + \ldots)$$

since, by this displacement $\Delta\xi$, the ordinate Y corresponding to any load will be increased by $\Delta\xi . \tan \alpha$. If the truss is directly loaded, so that the influence line is a curve instead of a polygon, the same expression applies provided that α_1, α_2 ... are the inclinations to the horizontal of the tangents to the curve at the load-points and $\Delta\xi$ is assumed to be an infinitesimal displacement. We thus find that to secure the most unfavorable loading (yielding the maximum value of the function) there must be a displacement either to the right or to the left according as the expression in the parenthesis is positive or negative in value; also that, in general, to secure the above condition one of the concentrations must rest at a panel point. The above expression may be represented graphically as follows: Lay off the loads upon a vertical line; commencing at the top of this load-line construct a polygon whose sides make the angles α_1, α_2 ... with the vertical and have for their vertical projections the respective forces P_1, P_2 .... The position of the terminal point of this

polygonal series of lines, to the right or left of the load-line, determines the direction of the required displacement of the series of concentrations.

Finally, the influence line also shows the effect of a continuous uniform load. This effect will be represented by the area included between the influence line and the axis of abscissae within the limits of the loading.

To obtain the *H-influence line or H-curve,* we must calculate the values of H by the preceding formulae for different possible positions of a unit load and plot these values as ordinates at the respective load-points. We may then find the values of H for any loading consisting either of a train of concentrations or of a continuous load. Thus, if we denote by $\eta_1, \eta_2 \ldots$ the ordinates of the H-curve at the points of application of the loads P_1, $P_2 \ldots\ldots$, then the horizontal force is given by

$$H = P_1\ \eta_1 + P_2\ \eta_2 + \ldots.$$

If, instead, we have a continuous load of intensity q per unit length distributed uniformly over the distance $x_2 - x_1$, then the horizontal force will be

$$H = q \int_{x_1}^{x_2} \eta\, d\,x = q.\Phi,$$

where Φ is the area of the H-curve included between the ordinates which mark the limits of the loading. If a hinge is introduced in the middle of the span, then, by equs. (64) and (65), the H-curve will be identical with the influence line for moments at the central section of the truss. With a simple (non-continuous) stiffening truss, the H-curve thus becomes a triangle having a maximum ordinate of $\frac{1}{4}\, G\, \frac{l}{f}$ at the crown.

Example. *a*) In a suspension bridge with a continuous stiffening truss, as in Fig. 27, let $l_1 = 0.4\ l$, $f = 0.1\ l$, $f_1 = 0.04\ l_1$, $I = 5\ I_1$ and $\frac{I}{A_0 f^2} = \frac{1}{60}$; also let $l_2 = 0.5\ l$, and $tan\ \alpha_1 = 0.25$. Then, by equ. (77), $\varepsilon = \frac{2\,(1 + (0.4)^3 . 5)}{3 + 2\,(0.4)\,.\,5} = 0.377$, and equs. (76) and (76^a) yield the following values of the horizontal tension:

Load in a side span:

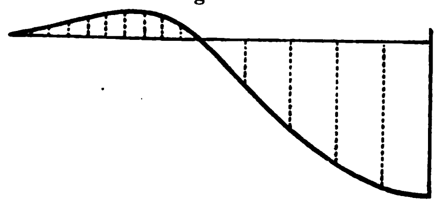
$\frac{\xi}{l_1} =$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$H =$	-0.0174	-0.0406	-0.0721	-0.1109	-0.1525	-0.1890	-0.2087	-0.19680	$-0.1343\ G$

Load in the main span:

$\frac{\xi}{l} =$	0.1	0.2	0.3	0.4	0.5
$H =$	0.7061	1.3832	1.9354	2.2944	2.4185 . G.

The accompanying Fig. 28 gives the H-curve constructed from the above values.

Fig. 28.



Example. **b)** If the stiffening truss is hinged at the towers, then, for the same data as in the preceding example, we obtain the following values of H by equs (79) and (79ᵃ):

For a load in a side span:

$$\frac{\xi}{l_1} = 0.1 \quad 0.2 \quad 0.3 \quad 0.4 \quad 0.5$$
$$H = 0.0671 \quad 0.1270 \quad 0.1739 \quad 0.2036 \quad 0.2139 \,.\, G.$$

For a load in the main span:

$$\frac{\xi}{l} = 0.1 \quad 0.2 \quad 0.3 \quad 0.4 \quad 0.5$$
$$H = 0.5245 \quad 0.9922 \quad 1.3587 \quad 1.5912 \quad 1.6710 \,.\, G.$$

According to the expression for H, the value of the horizontal force depends upon the assumed value of the ratio $\frac{I}{a_0 f^2}$. But, as previously noted, this ratio in all practical cases will fall between sufficiently close limiting values, and any error in estimating it will not affect the value of H appreciably. Thus, in the above example (a), if

$$\frac{I}{A_0 f^2} = \frac{1}{20}, \quad \frac{1}{80}, \quad 0, \quad \text{then } H \text{ will be}$$
$$0.85, \quad 1.021, \quad 1.093 \text{ times the values calculated above.}$$

§ 6. Values of *H* for Special Cases of Loading. The following formulae are derived by integrating the expressions derived above for the action of concentrated loads.

1. Stiffening Truss with Central Hinge. Single Span. If the truss is completely loaded with p per unit length, by (64),

$$H = \frac{1}{8} \cdot \frac{l}{f} \cdot p\,l \quad \text{.......... (80.}$$

If the truss is uniformly loaded for a distance λ from one end, by (64),

$$\left.\begin{aligned} &\text{for } \lambda < \frac{l}{2}, \quad H = \frac{1}{4}\,\frac{p\lambda^2}{f} \\ &\text{for } \lambda > \frac{l}{2}, \quad H = \frac{p}{4f}\left(2\,l\,\lambda - \lambda^2 - \frac{l^2}{2}\right) \end{aligned}\right\} \quad \text{...... (81.}$$

2. Stiffening Truss without Central Hinge; constant moment of inertia (I), parabolic cable.

a.) *Single Span.* If the truss is completely loaded with p per unit length, by (75),

$$H = \frac{1}{5 \cdot N_{75}} \cdot \frac{l}{f} \cdot p\,l \quad \text{.......... (82.}$$

If the truss is uniformly loaded for a distance λ from one end, by (75),

$$H = \frac{1}{N_{75}}\left[\frac{1}{5}\left(\frac{\lambda}{l}\right)^3 - \frac{1}{2}\left(\frac{\lambda}{l}\right)^2 + \frac{1}{2}\right]\left(\frac{\lambda}{l}\right)^2 \cdot \frac{l}{f} \cdot pl \quad \text{.. (83.}$$

Here N_{75} represents the denominator of equ. (75) or (75^a).

b.) *Three Spans. Stiffening Truss Continuous.* (Fig. 27.) Main span completely loaded: By (76),

$$\left.\begin{aligned} H = \frac{1}{N_{76}}\Big\{&\frac{1}{5} - \frac{1}{4}\,\varepsilon \\ &- \frac{1}{2}\,\frac{l\,I_1}{3\,l\,I_1 + 2\,l_1\,I}\left[1 - \frac{3}{2}\,\varepsilon + \left(\frac{f_1}{f} - \varepsilon\right)\frac{I\,l_1}{I_1\,l}\right]\Big\}\frac{l}{f} \cdot p\,l \end{aligned}\right\} \quad \text{...... (84.}$$

Both side spans completely loaded: By (76^a),

$$\left.\begin{aligned} H = \frac{2}{N_{76}}\Big\{&\frac{1}{5}\,\frac{f_1}{f} - \frac{1}{8}\,\varepsilon - \frac{1}{4}\,\frac{l\,I_1}{3\,l\,I_1 + 2\,l_1\,I}\left[1 - \frac{3}{2}\,\varepsilon\right. \\ &\left. + \left(\frac{f_1}{f} - \varepsilon\right)\frac{I\,l_1}{I_1\,l}\right]\frac{I_1}{I}\Big\}\frac{I\,l_1^2}{I_1\,l^2} \cdot \frac{l}{f} \cdot p\,l_1 \end{aligned}\right\} \quad \text{........ (85.}$$

Main span loaded for a distance λ from either tower: By (76),

$$\left.\begin{aligned}H=\frac{1}{N_{76}}\Big\{\Big[\frac{1}{5}\left(\frac{\lambda}{l}\right)^3-\frac{1}{2}\left(\frac{\lambda}{l}\right)^2+\frac{1}{2}\Big]-\Big(\frac{1}{4}\,\varepsilon\\+\frac{1}{2}\,\frac{l\,I_1}{3\,l\,I_1+2\,l_1 I}\Big[1-\frac{3}{2}\,\varepsilon+\left(\frac{f_1}{f}-\varepsilon\right)\frac{Il_1}{I_1 l}\Big]\Big)\\\left(3-2\frac{\lambda}{l}\right)\Big\}\left(\frac{\lambda}{l}\right)^2\cdot\frac{l}{f}\cdot p\,l\end{aligned}\right\}\ \ldots\ldots\ (86.$$

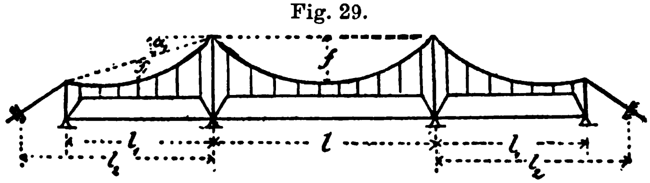
Side span loaded for a distance λ from the free end: By (76ᵃ),

$$\left.\begin{aligned}H=\frac{1}{N_{76}}\Big\{\Big[\frac{1}{5}\left(\frac{\lambda}{l_1}\right)^3-\frac{1}{2}\left(\frac{\lambda}{l_1}\right)^2+\frac{1}{2}\Big]\frac{f_1}{f}\\-\Big(\frac{1}{8}\,\varepsilon+\frac{1}{4}\,\frac{l\,I_1}{3\,l\,I_1+2\,l_1 I}\cdot\frac{I_1}{I}\Big[1-\frac{3}{2}\,\varepsilon+\left(\frac{f_1}{f}-\varepsilon\right)\frac{Il_1}{I_1 l}\Big]\Big)\\\left(2-\left(\frac{\lambda}{l_1}\right)^2\right)\Big\}\frac{I\,l_1^2}{I_1\,l^2}\cdot\frac{l}{f}\cdot p\,l_1\end{aligned}\right\}\ \ldots\ldots\ (87.$$

Here N_{76} represents the denominator of equs. (76) and (76ᵃ).

c.) *Three Spans. Stiffening Truss Interrupted (Hinged) at the Towers.* (Fig. 29.)

Fig. 29.



For loading in the main span, equs. (82) and (83) apply if there is substituted for N the denominator of equ. (79).

For a complete loading of both side spans, by (79ᵃ):

$$H=\frac{2}{5\,N_{79}}\cdot\frac{f_1}{f}\,\frac{Il_1^2}{I_1 l^2}\cdot\frac{l}{f}\cdot pl_1\ \ldots\ldots\ldots\ldots\ldots\ldots\ (88.$$

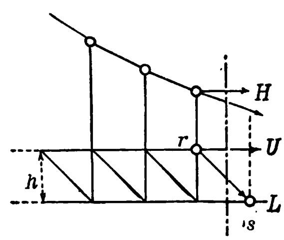
For a continuous load in either side span covering a distance λ from the free end, by (79ᵃ):

$$H=\frac{1}{N_{79}}\Big[\frac{1}{5}\left(\frac{\lambda}{l_1}\right)^3-\frac{1}{2}\left(\frac{\lambda}{l_1}\right)^2+\frac{1}{2}\Big]\left(\frac{\lambda}{l_1}\right)^2\cdot\frac{f_1}{f}\cdot\frac{Il_1^2}{I_1 l^2}\cdot\frac{l}{f}\,pl_1\ \ldots\ (89.$$

§ 7. Stresses in the Stiffening Truss.

1. To Find the Maximum Stresses in the individual members of the stiffening truss, it is necessary to first determine the maximum bending moments and shears in the structure. If it consists of a truss with parallel chords, the chord stresses are obtained from the bending moments and the web stresses from the shears. The moments M, according to equs. (54) to (58), may be represented by the products $H \cdot z$, where z denotes the vertical intercepts between the equilibrium polygon of the external loading and the axis of the cable. These moment intercepts must be measured in the vertical lines passing through the panel-points lying opposite the given chord members, and consequently (see Fig. 30),

Fig. 30.



$$\left.\begin{aligned} U_x &= -\frac{M_s}{h} = -\frac{z_s}{h} \cdot H \\ L_x &= +\frac{M_r}{h} = +\frac{z_r}{h} \cdot H \end{aligned}\right\} \qquad (90.$$

If, as in the system of Fig. 32, the horizontal thrust of the arch is taken up by the stiffening truss, then the resulting axial stress must naturally be taken into consideration in proportioning the members. The chord stresses will then be

$$\left.\begin{aligned} U_x &= -\frac{M_s}{h} + \frac{H}{2} = -\left(z_s - \frac{h}{2}\right) \cdot \frac{H}{h} \\ L_x &= +\frac{M_r}{h} + \frac{H}{2} = +\left(z_r + \frac{h}{2}\right) \cdot \frac{H}{h} \end{aligned}\right\} \qquad (91.$$

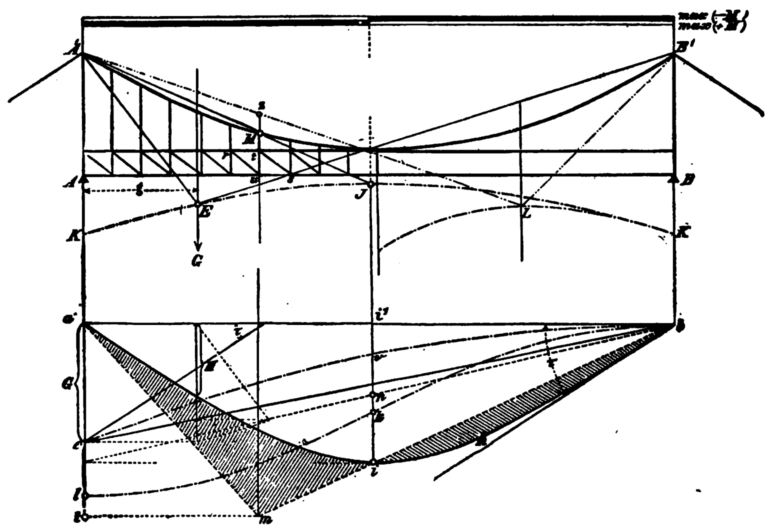
In general, therefore, we may put

$$\left.\begin{aligned} U_x &= -\frac{e_u}{h} \cdot H \\ L_x &= +\frac{e_l}{h} \cdot H \end{aligned}\right\} \qquad (92.$$

where e_u and e_l are the intercepts between the equilibrium polygon and the axis of the arch-curve, or (in the structures of Figs. 25 and 32) between the equilibrium polygon and two lines obtained by shifting the arch curve upward or downward by an amount $\frac{h}{2}$.

2. Reaction Locus. Loading for Maximum Stress. In order to determine the positions of the moving load for maximum stresses in the stiffening truss, it is necessary first to investigate the influence of a moving concentration. If the graphic method is selected for this purpose, the problem reduces to finding the equilibrium polygon for any position of the load. This involves

Fig. 31.



no special difficulty after the value of the horizontal thrust for the construction of the equilibrium polygon is obtained from the H-curve.

a.) *For a Simple Non-Continuous Stiffening Truss,* the equilibrium polygon must pass through the points of suspension $(A'B')$ of the cable. In this case the direction of the sides of the equilibrium polygon (or triangle) is simply found from the resultant of the horizontal tension H and the vertical end-reaction; this reaction, for a load placed at a distance ξ from the end A, is defined by $G\,\frac{l-\xi}{l}$, and hence is given by the ordinates of the straight line $b\ c$ (Fig. 31). If we construct the equilibrium polygon, as indicated in the figure, for other posi-

tions of the load, the geometric locus of the point E in which the two sides of the polygon intersect each other and the load-line will yield a curve $K\,E\,K$, which we name the *Reaction Locus*. The ordinates of this curve, measured from the chord $A'B'$, are analytically defined by

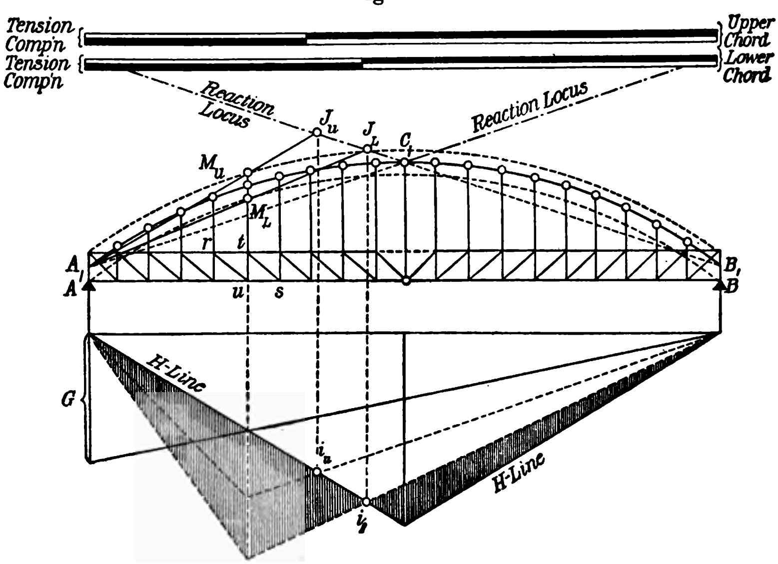
$$\eta = \frac{G(l-\xi)\xi}{H \,.\, l} \quad \text{(93.}$$

Substituting the approximate expression of (75[a]) for H, we have

$$\eta = \frac{N_{75} \,.\, l^2}{l^2 + l\xi - \xi^2} \cdot f \quad \text{(94.}$$

From the latter equation, by putting $\xi = 0$, we obtain the ordinate of the point K; to determine the same point by con-

Fig. 32.



struction we utilize the relation $A'\,K = G \,.\, cot\ a$, where a is the inclination of the tangent to the H-curve at a.

If the stiffening truss has an intermediate hinge, the moment at this section must equal zero; hence the equilibrium polygon must always pass through the point of the arch-curve directly above the hinge. The reaction locus in that case consists of two straight lines forming the prolongations of the chords $A_1\,C_1$ and $B_1\,C_1$ (Fig. 32).

The equations of these two lines, referred to $A_1\,B_1$ as axis of abscissae, are

$$\left.\begin{aligned}\eta &= \frac{2f}{l} \cdot (l-\xi) \\ \text{and} \qquad \eta &= \frac{2f}{l} \cdot \xi\end{aligned}\right\} \qquad (95.$$

Since, by equ. (90), the stress in the upper chord member $r\,t$ (Fig. 31), or in the lower chord member $s\,u$, is determined by the intercept between the equilibrium polygon and the point M of the cable, then the point J (Fig. 31), in which the connecting line $A'M$ intersects the reaction locus, will be the *critical point* for the above members, i. e., the position at which the moving concentration produces a change of stress in the given chord members. For all loads placed to the right of J, the equilibrium polygon will be above the cable-curve at M, hence z will be negative; while for all loads to the left of J (up to some point J_1 where the connecting line $B'M$ may cut the reaction locus), the equilibrium polygon will fall below the cable-point M, so that z will be positive. Consequently, a load covering the distance $J\,B$ will produce the *maximum tension in the upper chord* $r\,t$ and the *maximum compression in the lower chord* $s\,u$; while a load covering the remainder of the span $A\,J$ (or $J\,J_1$) will produce the *maximum compression in the upper chord* and the *maximum tension in the lower chord.*

For those systems in which the horizontal force H is taken up by the chords of the stiffening truss (Fig. 25), it is necessary, as deduced above from equ. (91), to replace the cable-curve by two similar curves distant $\pm \frac{h}{2}$ from it, the upper one governing the design of the upper chord and the lower one governing the design of the lower chord (Fig. 32).

Analytically, the abscissa of the load-division point or critical point J is given by

$$\xi_1 = \frac{x}{y} \cdot \eta_1 \qquad (96.$$

where x and y denote the coordinates of the point M referred to the closing chord A_1B_1. Substituting for η_1 its value from (94), we obtain the following equation for the bridge without a central hinge:

$$-\xi_1^3 + l\xi_1^2 + l^2\,\xi_1 = N_{75} \,.\, l^2 f \,.\, \frac{x}{y} \qquad (97.$$

For a center-hinged stiffening truss, by equ. 95, the critical point will be given by

$$\frac{\xi_1}{l} = \frac{2\,f\,x}{l\,.\,y + 2\,f\,x} \qquad (98.$$

and for a parabolic cable-curve,

$$\frac{\xi_1}{l} = \frac{l}{3l - 2x} \qquad (98^{a}.$$

b.) *In the Three-Span Suspension Bridge Having a Continuous Stiffening Truss,* there is an added difficulty; as the bending moments at the intermediate supports are no longer zero, the equilibrium polygon does not pass through the suspension-points of the cable. If $\mathbf{M}_1 = H.\mathbf{m}_1$ and $\mathbf{M}_2 = H.\mathbf{m}_2$ are the bending moments produced at the intermediate supports by the vertical loads acting on the continuous truss, then (by Fig. 21) the equilibrium polygon must pass through two points respectively $(\mathbf{m}_1 - m)$ and $(\mathbf{m}_2 - m)$ above the cable suspension-points, where m is determined by equs. (57) and (58). (This is equivalent to expressing the resultant bending moments at these points as the difference between the moments due to the downward-acting loads and those due to the upward-acting suspender forces.) The quantities $\mathbf{M}_1$ and $\mathbf{M}_2$, or $\mathbf{m}_1$ and $\mathbf{m}_2$, must be determined by the theory of the continuous beam (Theorem of Three Moments) either by computation or by construction:

In Fig. 33, the graphic solution is indicated. F, F_1 are two fixed points located in the main span by laying off $B_{,,}F = C_{,,}F_1$ $= l \div \left(3 + 2\frac{I\, l_1}{I_1\, l}\right)$. Then lay off $C_{,,}E = B_{,,}C_{,,} - 3F_1C_{,,}$ and $EG = BC \cdot \frac{F\,F_1}{B_{,,}F}$; also $GH = \frac{1}{H} \cdot G \cdot \frac{\xi\,(l-\xi)}{l}$. The last distance (GH) is equal to the intercept $M\,T$, at the point of loading, of the ordinary equilibrium polygon constructed with the pole distance H. The connecting line $E\,H$ intercepts on the load-vertical a distance $M\,L$; this is $\mathbf{m}_1$. (The corresponding analytical relation, derived from the Theorem of Three Moments, is

$$\mathbf{m}_1 = \left[\frac{G}{H} \cdot \frac{\xi\,(l-\xi)}{l}\right] \cdot \frac{2\,ir + \left(1 - \frac{\xi}{l}\right)(3 + 2\,ir)}{(1 + 2\,ir)\;(3 + 2\,ir)}$$

where r is the span ratio, $l_1 : l$, and i is the ratio of the moments of inertia, $I : I_1$.) Similarly, the distance intercepted by $E\,H$ on the symmetrically located vertical will be $\mathbf{m}_2$. Projecting these two intercepts ($\mathbf{m}_1$ and $\mathbf{m}_2$) upon the respective end-verticals, the resulting connecting line $N\,O$ will be the closing side of the equilibrium polygon; and transferring the altitude $M\,T$ to $P\,K$, we obtain $N\,K\,O$ as the true equilibrium polygon in its proper position relative to the cable-curve.

Going through this construction of the equilibrium polygon for different positions of a concentrated load, there is finally obtained the Reaction Locus $K_{,}K_{,,}$ as the geometric locus of the point K in which the sides of the equilibrium polygon, representing the end-reactions, intersect the load vertical. Furthermore, the different directions of the sides of the equilibrium polygon envelop a curve UU_1, which we call the *Reaction Tangent*

Curve. We also construct the curve $A_{,,} Q$ whose ordinates are the end-moments per unit $H(=\mathbf{m}_1)$, for different positions of the concentration in the side span. With a load at Q, if the point Q is projected horizontally to Q_1, the line drawn from Q_1 through the fixed point F_1 represents the line of the equilibrium polygon in the main span. (The corresponding analytical expressions are

$$B_{,,}Q_1 = -\mathrm{m}_1 = \left[\frac{G}{H}\cdot\frac{\xi(l_1-\xi)}{l_1}\right]\frac{2\,ir\,(1+ir)}{(3+2\,ir)\;(1+2\,ir)}\left(1+\frac{\xi}{l_1}\right)$$

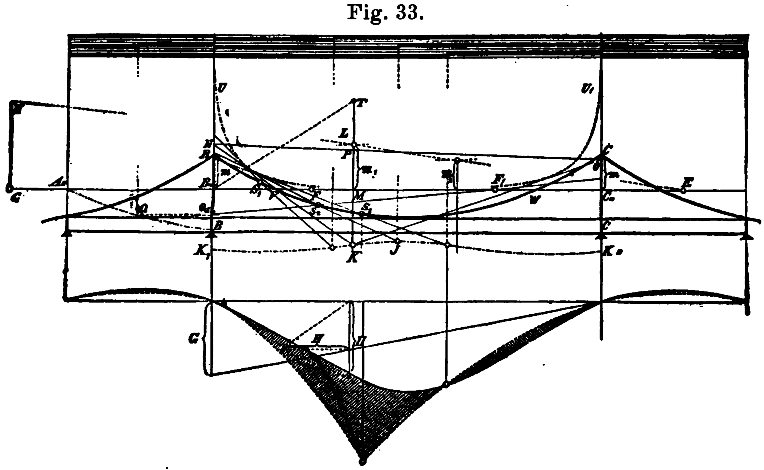
and

$$C_{,,}Q_{,,} = +\mathrm{m}_2 = \left[\frac{G}{H}\cdot\frac{\xi(l_1-\xi)}{l_1}\right]\frac{ir}{(3+2\,ir)\;(1+2\,ir)}\left(1+\frac{\xi}{l_1}\right)$$

where i and r have the significations given above.)

It now becomes a simple matter to determine the manner

Fig. 33.



of loading for producing the maximum bending moments at any section of the stiffening truss. If S_2 is the point of the cable at the given vertical section, then the tangent from S_2 to the Reaction Tangent Curve U will intersect the Reaction Locus at some point J which will be the "load-division point" (= critical point) in the main span; while the critical point in the left side span will be given by the point Q. In Fig. 33 are shown the diagrams of loading for three different cross-sections, S_1, S_2, S_3.

If the three-span bridge is hinged at the towers, the equilibrium polygons for a concentrated load and the resulting reaction locus are determined for each span exactly as in Fig. 31.

For producing maximum positive (or negative) moments at any section, the corresponding span must be loaded up to the critical point, and the other two spans must be free from (or full of) load.

3. Determination of the Moments for Full Loading and for the Severest (Critical) Loading.

a.) *Method of Influence Lines.* In a single-span bridge (Fig. 31), the moment produced at any cross-section M is expressed by

$$M = \mathrm{M} - Hy = y\left[\frac{\mathrm{M}}{y} - H\right] \quad \ldots\ldots\ldots\ldots\ldots (99.$$

For a moving concentration, $\frac{\mathrm{M}}{y}$ represents the moment influence line of a simple beam, constructed with the pole distance y. Hence the moment M is proportional to the difference between the ordinates of this influence line and those of the H-curve.

The influence line for the simple-beam moments $\frac{\mathrm{M}}{y}$ is familiarly obtained as a triangle whose altitude at the given section $= \frac{x\,(l-x)}{l \cdot y}$. It may be constructed, however, by using the point J (Figs. 31 and 32), which is obtained by the aid of the reaction locus as previously described, and projected down to the point i in the H-curve. If Φ_1 and Φ_2 are the areas respectively above and below the H-curve, included between it and the above-mentioned influence line (areas $i\,b\,h$ and $a\,m\,i$, in Fig. 31), then, for a uniformly distributed load of p per unit length,

$$\left.\begin{aligned} max\,(+M) &= \frac{p}{G} \cdot \Phi_2 \cdot y \\ max\,(-M) &= \frac{p}{G} \cdot \Phi_1 \cdot y \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (100.$$

For finding the areas Φ, proceed as follows: First determine the areas of the H-curve by analytic or graphic integration, divide by $\frac{l}{2}$, and plot the resulting lengths as the ordinates of a curve $b\,k\,l$; so that, for instance, the area $b\,i'\,i\,h = \overline{i'k} \cdot \frac{l}{2}$ (Fig. 31). Then the ordinate $i'k$ gives the value of H for a uniform load covering the length $B\,J$ if, at the same time, $\overline{a\,c} = G = \frac{p\,l}{2}$. Similarly, dividing the area of the triangle $i\,i'b$ by $\frac{l}{2}$ gives the ordinate $i'n$. Consequently we have

$\Phi_1 = \overline{kn} \cdot \frac{l}{2}$ and

$$max\,(-M) = \overline{kn}\,.\,y.$$

In the same manner we find the effect of a full loading to be

$$M_{tot} = \overline{tl}\,.\,y,$$

and therefore the maximum positive moment will be

$$max\,(+M) = (\overline{kn} + \overline{tl})\,.\,y.$$

The distances $\overline{kn}$ and $\overline{tl}$ are to be measured on a force-scale fixed by the relation $\overline{ac} = \frac{pl}{2}$.

The above treatment may also be applied to the continuous stiffening truss (Fig. 33), but in that case the influence line for the moments **M** is no longer simply a triangle but must be specially derived from the theory of the continuous beam. For this purpose we may utilize the equilibrium polygons which were required in constructing the reaction locus; the ordinates of these polygons measured below the horizontal line $B_{,,}$ $C_{,,}$ and divided by $(y - m)$ give the corresponding ordinates of the required influence curve. At the sections near the points V and W, where the values of $y - m$ are very small, this method is not conveniently applicable; but, for these sections, the moment M is very nearly (and for the points V and W exactly) equal to the moments **M** of an ordinary continuous beam.

b.) *Determination of the Maximum Moments from the Equilibrium Polygon of the Loading.* If z denotes the intercept between the equilibrium polygon of the loading and the axis of the cable or arch, measured in the vertical line through the section M, and if **H** is the horizontal force, then, by the principle established in § 5, the moment in the stiffening truss at the section M is given by

$$M = \mathbf{H}\,.\,z \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (101.$$

The quantities **H** and z may be determined either by computation or by construction. In the latter case, we must first find the value of **H** for the given loading from the ordinates or areas of the H-influence line and, with this as a pole distance, construct the equilibrium polygon of the loading in its proper position relative to the cable-curve.

For uniformly distributed loads, the expressions for the horizontal force **H** have been deduced in § 6. These expressions may be plotted to give an **H**-curve which, as previously mentioned, may also be constructed by graphic integration of the

ordinary H-curve. If, in addition, we construct, for a single-span stiffening truss, the parabola $b\,v\,c$ as the integrated influence line for vertical reactions (Fig. 31), then the two intercepts $i'v$ and $i'k$ will represent respectively the horizontal and vertical components of the left end-reaction for a load covering the section $J\,B$ of the truss; and the vector resultant of these two forces will give the direction of the side of the equilibrium polygon passing through the suspension-point A'. The distance Mz intercepted by this line on the vertical through M is the required factor z. The determination of this quantity may be simplified by drawing the locus of the point L in which the polygon-side $A'L$ intersects the center-line of the loaded segment; the resulting curve is called the *Second Reaction Locus.*—We then obtain the maximum negative moment at the section M by multiplying together the two distances $i'k$ and Mz, measuring the first by the scale of forces $\left(\overline{ac}=\frac{p\,l}{2}\right)$ and the second by the scale of lengths.

Assuming a parabolic cable-curve, we may establish the following analytical expressions with the aid of equations (80) to (89):

1. *Three-Hinged Stiffening Truss.* (*Single Span with Central Hinge*). Under complete loading, the equilibrium curve coincides throughout with the cable-curve. Hence the load is carried entirely by the cable and the stiffening truss remains unstressed, i. e., $M_{tot}=0$.

Under the critical loading defined by equ. 98ª, we find

$$M_{\substack{max\\min}}=\pm\frac{p\,x\,(l-x)\,(l-2\,x)}{2\,(3\,l-2\,x)}\;\ldots\ldots\ldots\ldots\ldots \quad (102.$$

From this expression we find that the absolute maximum moment will occur at $x=0.234\,l$ and will have the value

$$\textit{Absolute max.}\ M=\pm\,0.01883\;pl^2\;\ldots\ldots\ldots \quad (102^a.$$

If one-half of the span is loaded, there will be produced positive moments in the loaded half and negative moments in the unloaded half amounting to $M=\frac{1}{4}\,px\left(\frac{l}{2}-x\right)$; and the maximum moment, occurring at the quarter-point, will be

$$\textit{Max.}\ M=\frac{1}{64}\,p\,l^2=0.01562\,p\,l^2.$$

2. *Stiffening Truss without Central Hinge; Parabolic Cable, Constant I for Truss.*

a. Single Span. With the span completely loaded, the bending moment at any section will be $M=\frac{1}{2}\;p\,x\;(l-x)\;-\;\mathbf{H}\,.\,y$;

substituting the value of **H** from equ. (82), we obtain

$$M_{tot} = \frac{1}{2} p\, x\, (l - x) \frac{5 N_{75} - 8}{5 N_{75}} \ldots\ldots\ldots\ldots \text{(103.}$$

For the critical loading, applying equs. (83) and (97), we obtain

$$M_{min} = -\frac{2 p x (l-x)}{5 N_{75}} \left[2 - \left(\frac{\xi_1}{l}\right) - 4\left(\frac{\xi_1}{l}\right)^2 + 3\left(\frac{\xi_1}{l}\right)^3\right]\left(1 - \frac{\xi_1}{l}\right)^2 . \text{(104.}$$

From this value the maximum moment may be derived by the relation

$$M_{max} = M_{tot} - M_{min} \ldots\ldots\ldots\ldots\ldots\ldots \text{(104}^a\text{.}$$

In equ. (104) the value of ξ_1 must be obtained from equ. (97). N_{75} denotes the denominator of the expression for H in equ. (75). The absolute maximum moment is affected by the value of N_{75}; substituting its smallest value, $N_{75} = \frac{8}{5}$, corresponding to a very weak stiffening truss, we obtain the following value of the absolute maximum moment, occurring at $x = 0.250\, l$:

$$\textit{Absol. max. } M = \pm\, 0.01652\, pl^2 \ldots\ldots\ldots\ldots \text{(104}^b\text{.}$$

With the half-span loaded, we find in this case, as in the preceding, max. $M = \frac{1}{64}\, pl^2$; this moment occurs, as before, at the quarter points.

As the quantity N, representing the stiffness of the truss, increases in value, the maximum negative moments diminish, whereas the full-load moments and the maximum positive moments are increased with respect to the above values.

Equation (104) applies to all sections from $x = 0$ to $x_1 = \frac{l}{4} \cdot N_{75}$. For the minimum moments at the sections near the center, from x_1 to $l - x_1$, it is necessary to bring on some load also from the left end of the span; so that the expression for these moments consists of two parts which may be obtained by applying equ. (104) to the two symmetrically located points x and $l - x$.

In Figs. 34a and 34b are plotted the graphs representing the maximum moments in suspension bridges with and without a central hinge.

For the second graph there was selected the example given on page 37. The data were $l = 50$ meters, $f = 6.5$ m., $l_1 \sec^2 a_1 = 19.05$ m., $\frac{I}{A_0} = 2.049$, $A_0 = 0.03$ sq. meters; hence $N = 1.8692$, giving a moment at the center of the span, for a full-span load, equal to $0.0180\, pl^2$.

β. Three Spans. Stiffening Truss Continuous. (Fig. 27.) With all three spans loaded, the moment at any section of the main span is given by

$$M_{\text{tot}}=\left[\frac{1}{2}p-\mathbf{H}\frac{4f}{l^2}\right]x\,(l-x)-\varepsilon\left(\frac{1}{8}p\,l^2-\mathbf{H}\,f\right)\,.. \quad (105.$$

and at any section of the side span distant x from the free end

$$M_{\text{tot}}=\left(\frac{1}{2}p-\mathbf{H}\frac{4f_1}{l_1^2}\right)x\,(l_1-x)-\varepsilon\left(\frac{1}{8}p\,l^2-\mathbf{H}\,f\right)\frac{x}{l_1}\,.. \quad (106.$$

where the value of ε must be taken from equ. (77) and that of $\mathbf{H}$ from the combination of equs. (84) and (85).

The maximum moments, produced by the critical loading, must be calculated by the general equs. (58ᵃ).

Fig. 34a.
Maximum Moments in Stiffening Truss with Central Hinge.

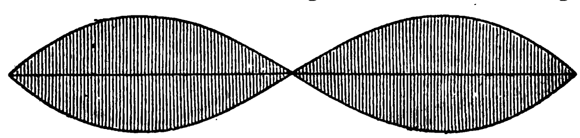
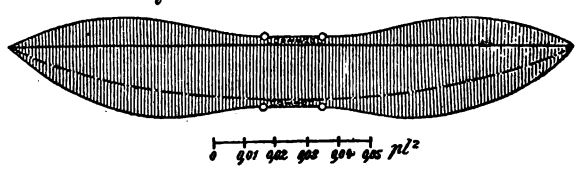


Fig. 34b.
Maximum Moments in Stiffening Truss without Central Hinge.

$\frac{f}{l}=0.13, \quad N=1.8692$



γ. Three Spans. Stiffening Truss Hinged at the Towers. (Fig. 29.) With all three spans loaded, we have

$$M_{\text{tot}}=\frac{1}{2}p\,x\,(l-x)\left[1-\frac{8}{5\,N_{79}}\left(1+2\frac{f_1}{f}\,\frac{I\,l_1^3}{I_1\,l^3}\right)\right]\,.. \quad (107.$$

or, since $N_{79}=N_{75}\left(1+2\,\frac{f_1}{f}\,\frac{I\,l_1^3}{I_1\,l^3}\right)$ very nearly, we may also apply equ. (103) to this case, the moments agreeing very closely with those of a single-span bridge, completely loaded. The moments in the side spans are also given by equ. (103) if l_1 is written instead of l.

The maximum negative moment at any section of the main span is obtained by loading a length $l-\xi_1$ in that span and completely loading both side spans. Then, by (83) and (88),

$$M_{\min}=-\frac{2px\,(l-x)}{5\,N_{79}}\left\{\left[2-\left(\frac{\xi_1}{l}\right)-4\left(\frac{\xi_1}{l}\right)^2+3\left(\frac{\xi_1}{l}\right)^3\right]\left(1-\frac{\xi_1}{l}\right)^2+4\frac{I\,l_1^3 f_1}{I_1\,l^3 f}\right\}\,.. \quad (108$$

The maximum negative moment at any section of a side span is obtained by loading a length $l_1 - \xi_1$ in that span and completely loading both the other spans. Then, by (82), (88), (89),

$$M_{min} = -\frac{2px(l_1-x)}{5N_{79}}\left\{\begin{aligned}&\left[2-\frac{\xi_1}{l_1}-4\left(\frac{\xi_1}{l_1}\right)^2+3\left(\frac{\xi_1}{l_1}\right)^3\right]\\ &\left(1-\frac{\xi_1}{l_1}\right)^2+\frac{2f_1l^2}{fl_1^2}\left(1+\frac{I\,l_1^3f_1}{I_1l^3f}\right)\end{aligned}\right\} \quad \ldots (108^a.$$

For the main span, in equ. (108), the value of ξ_1 is defined by the relation similar to equ. (97),

$$-\xi_1^3 + l\,\xi_1^2 + l^2\,\xi_1 = N_{79}\,l^2 f\cdot\frac{x}{y};$$

and for the side span, in equ. (108^a), the value of ξ_1 is defined by the following relation obtained from equs. (79^a), (93) and (96),

$$(-\xi_1^3 + l_1\,\xi_1^2 + l_1^2\,\xi_1)\,\frac{I\,l_1 f_1^2}{I_1\,l\,f^2} = N_{79}.\,l_1^2 f_1\,\frac{x}{y}\,.$$

The maximum positive moments are given by the relation

$$M_{max} = M_{tot} - M_{min}{}^2$$

4. Determination of the Maximum Shears. The critical loading for shears at any section may be obtained from a consideration of equ. (59). Accordingly, the load-point yielding a change in sign of the transverse shear is found by locating the point J (Fig. 35) in which the reaction locus is cut by a polygon-ray drawn at an angle τ, i. e., parallel to the cable-tangent at the given section. In addition, the sign of S changes at the section itself, so that the critical loading is as indicated in Fig. 35. The same rule applies to the main span of the continuous form of suspension bridge but, at the same time, the side span nearer the section must be completely loaded and the other side span free from load or partially loaded for a maximum positive shear.

To determine the maximum shears we may again employ the two methods previously introduced:

a.) *Method of Influence Lines.* The shear at any section of the stiffening truss is given by equ. (59) as

$$S = \mathbf{S} - H\ (\tan\ \tau - \tan\ \sigma)$$

$$= (\tan\tau - \tan\sigma)\left[\frac{\mathbf{S}}{\tan\tau - \tan\sigma} - H\right].$$

Here σ denotes the inclination of the connecting line $A'\,B'$ below the horizontal (+ if A' is higher than B') (cf. Fig. 20). If the stiffening truss is one with parallel chords, and if δ is the

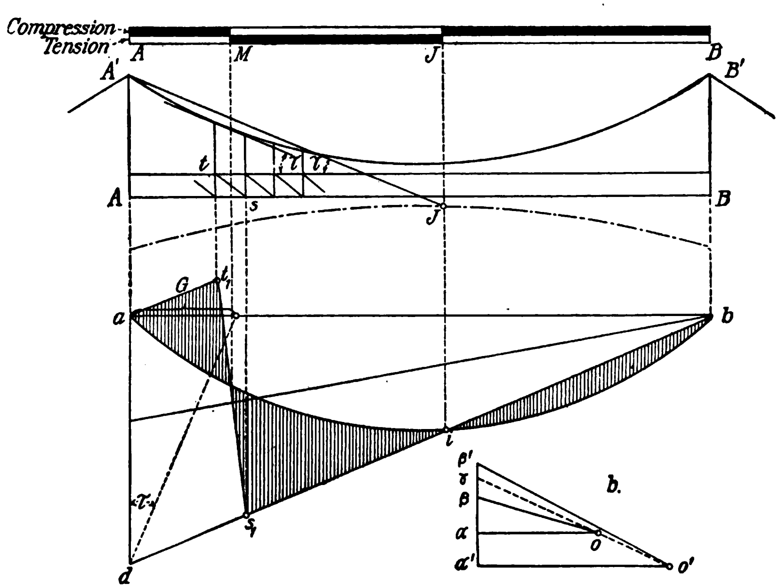
inclination to the vertical of the web member at the given section, then the stress in this web member will be

$$P = S \,.\, sec\,\delta = sec\,\delta \,.\, (tan\,\tau - tan\,\sigma)\left[\frac{\mathbf{S}}{tan\,\tau - tan\,\sigma} - H\right] \quad ..(109.$$

The values assumed by the bracketed expression for different positions of a concentrated load may be easily represented. They are, namely, the differences between the ordinates of the H-curve and those of the influence line for the shears **S**, the latter being reduced in the ratio of $\frac{1}{tan\,\tau - tan\,\sigma}$ or, if the

Fig. 35.

a



suspension points are at the same level, in the ratio of $cot\,\tau : 1$. The latter influence line is familiarly obtained by drawing the two parallel lines $a\,t_1$ and $b\,s_1$ (Fig. 35), their direction being fixed by the intercept $a\,d = G \;\frac{1}{tan\,\tau - tan\,\sigma}$ laid off on the end vertical. The vertices t_1 and s_1 lie on the verticals passing through the panel-points of the given web member. The intersection i with the H-curve, which should fall directly under the critical point J, affords a check on the construction. The maximum shears or shearing stresses produced by a uniformly

distributed load are determined by the areas included between the H and **S** influence lines; all areas below the H-curve are to be considered positive and all above negative. These areas must be multiplied by the coefficient $\frac{p}{G}$ *sec* δ (*tan* τ — *tan* σ) to obtain the greatest stresses in the web member.

Equ. (109) is applicable also to the stiffening truss continuous over four supports, if $\sigma = 0$ and the structure is symmetrical about the center line; but in this case the S-influence line no longer consists of straight lines.

At the sections near the center line, where $\tau = o$, the shears are very nearly the same as in an ordinary truss, i. e., $S = \mathbf{S}$ very nearly.

b.) *Determination of the Shears by Means of the Equilibrium Polygon for Partial Loading.* With the aid of the second reaction locus, we may again construct the equilibrium polygon and the corresponding force polygon for partial loading. (Fig. 35*b*.) In the latter diagram, $\overline{o\ \alpha\ \beta}$ is the force polygon for a load covering the length $J\ B$: $\overline{o\ \alpha} = H$, $\overline{o\ \beta}$ is parallel to the polygon-side at A', hence $\overline{\alpha\,\beta} = \mathbf{S}$, and if $\overline{o\,\gamma}$ is drawn at an angle τ, $\overline{\gamma\,\beta} = S$ for the load $J\ B$; similarly $\overline{o'\ \alpha'\ \beta'}$ is the force polygon for the load $M\,B$, so that $\overline{\gamma\,\beta'}$ is the corresponding value of S; the difference between the two shears S, represented by $\overline{\beta\,\beta'}$, is the maximum positive shear $(+ S_{max})$ for the section M.

On the assumption of a parabolic cable-curve, we may develop the following expressions for the maximum shears:

1. *Stiffening Truss with Central Hinge. Single Span.* With the span completely loaded, the equilibrium curve for the loading coincides with the curve of the cable; hence the shears will be $S_{tot} = 0$.

For the maximum positive shear, the truss must be loaded from the given section to a point whose abscissa is $\xi_2 = \frac{l^2}{3l-4x}$. For $x > 0\,.\,25\,l$, the load must extend from the section to the end of the truss. We thus obtain, for $x = 0$ to $0\,.\,25\,l$,

$$\left.\begin{aligned} &max.\,(\pm S) = \pm \frac{p(l^2 - 3lx + 4x^2)^2}{2l^2\,(3l - 4x)} \\ &\text{for } x = 0.25 \text{ to } 0.5\,l, \\ &max.\,(\pm S) = \pm \frac{px^2\,(3l - 4x)}{2l^2} \end{aligned}\right\} \quad \ldots\ldots.(110.$$

The absolute maximum shears occur at the supports and at

the center of the span and amount to *max.* $(\pm S) = \frac{1}{6}\, pl$ and $\frac{1}{8}\, pl$ respectively.

2. *Stiffening Truss without a Central Hinge. Constant Moment of Inertia.*

a.) *Single Span.* With complete loading, the shear at a distance x from the end of the span will be, by (59) and (82),

$$S_{tot} = \frac{1}{2}\, p\, (l - 2x)\, \frac{5\, N_{75} - 8}{5\, N_{75}} \quad \ldots\ldots\ldots\ldots (111.$$

Loading the truss from the given section to the end of the span, the maximum positive shear will be, by (59) and (83),

$$S_{max} = \frac{1}{2}\, pl\left(1 - \frac{x}{l}\right)^2 \left\{1 - \frac{8}{N_{75}}\left(1 - \frac{2x}{l}\right)\left[\frac{1}{5}\left(1 - \frac{x}{l}\right)^3 - \frac{1}{2}\left(1 - \frac{x}{l}\right)^2 + \frac{1}{2}\right]\right\} \ldots (112.$$

For the sections near the ends of the span, from $x = 0$ to $x = \frac{l}{2}\left(1 - \frac{N_{75}}{4}\right)$, the loads must not extend to the end of the span to give the maximum positive shears, but must extend only to a point whose abscissa, ξ_2, is determined by the following equation, deduced from (94):

$$\xi_2(l^2 + l\xi_2 - \xi_2^2) = \frac{N_{75}}{4}\, \frac{l^4}{l - 2x} \quad \ldots\ldots\ldots\ldots (113.$$

For these sections, the shears given by equ. (112) must be increased by an amount

$$\frac{1}{2}\, pl\left(1 - \frac{\xi_2}{l}\right)^2 \left\{\frac{8}{N_{75}}\left(1 - \frac{2x}{l}\right)\left[\frac{1}{5}\left(1 - \frac{\xi_2}{l}\right)^3 - \frac{1}{2}\left(1 - \frac{\xi_2}{l}\right)^2 + \frac{1}{2}\right] - 1\right\} \quad (112^a.$$

We have also:

$$S_{min} = S_{tot} - S_{max}.$$

The largest shear occurs at the abutments; with the approximation $N_{75} = \frac{8}{5}$, its value is

Absolute max. $(+S) = 0.1523\, pl.$

At the center of the span,

$$max.\ (+S) = \pm \frac{1}{8}\, pl.$$

In Figs. (36a) and (36b) are shown the graphs of the maximum shears.

β.) Three Spans. Stiffening Truss Hinged at the Towers. (Fig. 29.) With the three spans completely loaded, the shear at any section in the main span will be, by (82), (88) and (59),

$$S_{tot} = \frac{1}{2} p\,(l - 2x)\left[1 - \frac{8}{5\,N_{79}}\left(1 + 2\frac{I l_1^3 f_1}{I_1 l^3 f}\right)\right] \quad \ldots . \ (113^a.$$

and in the side spans,

$$S_{tot} = \frac{1}{2} p\,(l_1 - 2x)\left[1 - \frac{8}{5\,N_{79}}\,\frac{f_1 l^2}{f l_1^2}\left(1 + 2\frac{I l_1^3 f_1}{I_1 l^3 f}\right)\right] \quad . . \ (114.$$

The maximum positive shears in the main span are calculated, exactly as for a single span, by equs. (112) and (112ª)

Fig. 36a.

Maximum Shears for Stiffening Truss with Central Hinge.

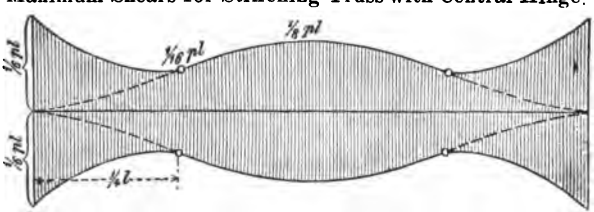
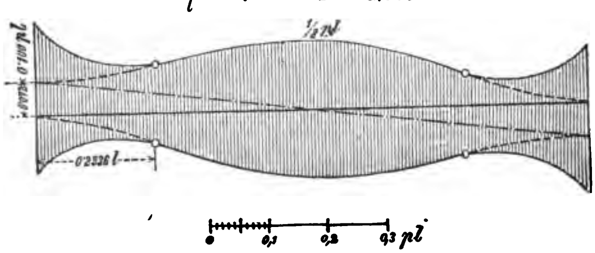


Fig. 36b.

Maximum Shears for Stiffening Truss without Central Hinge.

$\frac{f}{l} = 0.13 \qquad N = 1.8692$



but substituting for N_{75} the denomination of equ. (79). In the side spans,

$$S_{max} = \frac{1}{2} p l_1 \left(1 - \frac{x}{l_1}\right)^2 \left\{1 - \frac{8}{N_{79}} \cdot \frac{I l_1 f_1^2}{I_1 l f^2}\left(1 - \frac{2x}{l_1}\right)\left[\frac{1}{5}\left(1 - \frac{x}{l_1}\right)^3 - \frac{1}{2}\left(1 - \frac{x}{l_1}\right)^2 + \frac{1}{2}\right]\right\} \quad . \ (115.$$

and for the sections near the ends of these spans, namely from $x = 0$ to $x = \frac{l_1}{2}\left(1 - \frac{1}{4} N_{79} \frac{I_1 l f^2}{I l_1 f_1^2}\right)$ and from $x = \frac{l_1}{2}\left(1 + \frac{1}{4} N_{79} \frac{I_1 l f^2}{I l_1 f_1^2}\right)$ to $x = l_1$, the value given by equ. (115) for the maximum shear must be increased by the amount

$$\frac{1}{2}pl_1\left(1-\frac{\xi_2}{l_1}\right)^2\left\{\frac{8}{N_{79}}\frac{Il_1f_1^2}{I_1lf^2}\left(1-\frac{2x}{l_1}\right)\left[\frac{1}{5}\left(1-\frac{\xi_2}{l_1}\right)^3 -\frac{1}{2}\left(1-\frac{\xi_2}{l_1}\right)^2+\frac{1}{2}\right]-1\right\} \quad (116.$$

where ξ_2, the abscissa of the second critical point, must be obtained from

$$\xi_2\,(l_1^2+l_1\,\xi_2-\xi_2^2)=\frac{1}{4}N_{79}\cdot\frac{I_1lf^2}{Il_1f_1^2}\cdot\frac{l_1^4}{l_1-2x}\quad\ldots\ldots(117.$$

The above equations (115), (116) and (117) for the side spans are analogous in form and derivation to equs. (112), (112[a]) and (113) for the main span.

5. Effect of Temperature Variations.—If t is the variation of temperature from the condition of no stress and ω is the coefficient of linear expansion, then the equation of condition for the horizontal force H to replace equ. (67) will be

$$\int\left(\frac{N}{EA}+\omega t\right)\frac{dN}{dH}\cdot ds+\int\frac{M}{EI}\cdot\frac{dM}{dH}\cdot ds=0\ldots(118.$$

If the denominators of the expressions in (69), (70), etc., are denoted by N_{69}, N_{70}, etc., and if $\Sigma\frac{\lambda^2}{a}+2l_1\,sec^2\,\alpha_1$ is assumed approximately equal to the cable-length L, then the development of the above equation in the same manner as on page (32) will yield the following expressions for H_t for the several cases:

a.) *Single Span.* Stiffening truss without a central hinge, cases a) and b), (p. 31):

$$\left.\begin{aligned}H_t&=-\frac{\left[L+\Sigma y'\frac{\Delta^2y}{a}-2f'\,tan\,\alpha\right]A_0.E.\omega.t}{N_{69}}\\ H_t&=-\frac{\left[L+\Sigma y'\frac{\Delta^2y}{a}+2f'(tan\,\alpha+tan\,\alpha_1)\right]A_0.E.\omega.t}{N_{70}}\end{aligned}\right\}.\ (119.$$

and, neglecting the effect of the suspension rods,

$$H_t=-\frac{L.E.\omega.t}{N_{71}}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(120.$$

In the type of structure shown in Fig. 25, a uniform change of temperature in all the members will produce no stresses in the structure. But, in that construction, if the arch undergoes a temperature change of t and the stiffening truss and rods are subjected to a temperature variation of t', the horizontal stress will be

$$H_t = + \frac{\left(l - \Sigma \frac{y}{a}\, \Delta^2 y\right) A_0 E \,.\, \omega\,(t - t')}{N_{69} + \frac{A_0}{2 A_1} \cdot l} \Bigg\} \ldots\ldots (121.$$

For determining the moments and shears we have the equations

$$M = - H_t \,.\, y\,, \qquad S = - H_t \,.\, tan\, \tau.$$

In a similar manner we may obtain the effect of a displacement of the points of support of the cable produced either by a yielding of the anchorages or by a stretching of the backstays. If the span-length is thus diminished by an amount $\delta\, l$, then the variation in the horizontal tension will be

$$\Delta H = - \frac{E \,.\, \delta\, l}{N_{71}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (122.$$

With the approximate assumption of a continuously curved parabolic cable, the denominators of the expressions (119), (120) and (122) may be replaced by $\frac{1}{3} \frac{f^2 l}{I} \cdot N_{75}$ (where N_{75} represents the denominator of expression (75) or (75^a)). Neglecting the effect of the suspension rods there results,

$$H_t = - \frac{15}{8} E\, \omega\, t \frac{I}{f^2} \cdot \frac{L}{l} \cdot \frac{1}{1 + \frac{15}{8} \frac{I}{A_0 f^2} \frac{L}{l}} \ldots . (123.$$

or, substituting the value of $\frac{L}{l}$,

$$H_t = - \frac{\left(1 + \frac{16}{3} \frac{f^2}{l^2} + \frac{2 l_1}{l} sec^2\, a_1\right) . E . \omega . t . I}{\frac{8}{15} f^2 + \frac{I}{A_0} \cdot \left(1 + \frac{16}{3} \frac{f^2}{l^2} + 2 \frac{l_1}{l} sec^2\, a_1\right)} \ldots . (123^a$$

In the example, for which the maximum live-load moments are plotted in Fig. 34b, the horizontal tension due to a temperature variation of $\pm$ 30° is computed by equ. (123^a) as follows:

$$H_t = \frac{(1.8521)\ (20{,}000{,}000)\ (0.0000118)\ (30)\ (0.06147)}{24.3854} = \mp 33.05 \text{ tonnes.}$$

The resulting moment at the center of the stiffening truss attains a value of $\pm$ 214.8 t. m.

b.) *In the Three-Span, Continuous Suspension Bridge,* the horizontal tension due to a uniform variation of temperature throughout the structure will be:

$$H_t = - \frac{3\, E\, I\, \omega\, t \left[\left(1 + \frac{16}{3} \frac{f^2}{l^2}\right) + 2 \frac{l_1}{l} \left(1 + \frac{16}{3} \frac{f_1^2}{l_1^2} + tan^2 a_1\right)\right]}{f^2 . N_{76}} \ldots . (124.$$

If the stiffening truss has a central hinge, it might be sup-

posed that no temperature stresses can occur, since, in a statically determinate structure, no change can be produced in the horizontal force except by the application of external forces to the system. Actually, however, there will be a change of stress since, in general, the deformation of the cable accompanying the sag or rise at the central hinge cannot be neglected from consideration; so that the original assumption underlying the approximate theory can no longer be retained. Engineer G. Lindenthal was the first to call attention to this point:* but his approximate method, too inaccurate for a very stiff truss, results in overestimating the effect of temperature in the structure with a central hinge.

When the cable is lengthened by a temperature rise of t° (or by a displacement of the cable supports), the crown or central hinge will sag through some distance Δf. If we assume that the axes of the two halves of the truss remain straight during this sag, there will result a break in the cable-curve at the mid-point; hence the cable will no longer conform to the parabolic equilibrium curve corresponding to a uniformly distributed load. There must therefore result a change in the distribution of the suspender-forces: the suspension rods near the ends A and B, as well as those near the central hinge, C, will increase in stress while the intermediate rods will be relieved of load. The effect of this is to give the stiffening truss an increased share of the load at the intermediate points, causing it to deflect downward. The reverse occurs with a rise of the cable crown: the rods at the points specified above are subjected to increased stress and the truss is bent upward.

A rigorous treatment of the problem,† along the main lines developed in § 9, leads to the following results: Let H be the horizontal tension in the cable for that condition in which the cable coincides with the equilibrium curve of the external loading, so that no forces are acting on the stiffening truss. As a result of temperature variations or other cause, let the cable-crown deflect downward through the distance Δf. Let the accompanying deflection of the truss at a distance x from the abutment be η. Then, neglecting the elongation of the suspenders and denoting the half-span by a, the cable ordinate is changed into $y + \frac{x}{a} \cdot \Delta f + \eta$

and the horizontal tension becomes

$$H' = H \cdot \frac{f}{f + \Delta f} \cdot$$

* Report of Board of Engineer Officers, Washington, 1894. Appendix D: Temperature Strains in Three-Hinged Arches.

† *Melan,* Die Ermittelung der Spannungen im Dreigelenkbogen und in dem durch einen Balken mit Mittelgelenk versteiften Hängeträger mit Berücksichtigung seiner Formänderung. Österr. Wochenschr. f. d. öffentl. Baudienst. 1903, No. 28.

The bending moment in the stiffening truss will then be

$$M = (H - H')\, y - H' \left(\frac{x}{a} \Delta f + \eta\right)$$

or

$$M = \left[\left(y - \frac{x}{a} f\right) \Delta f - \eta f\right] \frac{H'}{f}.$$

If the cable is initially a parabola, then

$$y = \frac{x\,(2\,a - x)\,.\,f}{a^2},$$

hence,

$$M = \left[\frac{x\,(a - x)}{a^2} \cdot \Delta f - \eta\right] H'.$$

The differential equation of the elastic curve of the truss, $E\,I\, \frac{d^2 \eta}{d\,x^2} = -M$, on introducing the abbreviation

$$c^2 = \frac{H'}{E\,I} \div \frac{H}{E\,I} \text{ .... } (125.$$

becomes

$$\frac{d^2 \eta}{d\,x^2} - c^2 \eta + c^2 \frac{x\,(a - x)}{a^2} \cdot \Delta f = 0.$$

The integration of this equation finally yields the following expression for the moments in the stiffening truss:

$$M = \left[1 - \frac{e^{cx} + e^{c(a-x)}}{1 + e^{ca}}\right] 2\,E\,I \cdot \frac{\Delta f}{a^2} \text{ .... } (126.$$

The maximum value, at $x = \frac{a}{2}$, will be

$$M_{\max} = \frac{(1 - e^{\frac{1}{2}ca})^2}{1 + e^{ca}} \cdot 2\,E\,I \cdot \frac{\Delta f}{a^2} \text{ .... } (127.$$

The bending moments are therefore directly proportional to the crown deflection Δf; but not exactly to the quantity I, because this is also involved in the coefficient c. Only for small values of I is M proportional to the stiffness of the truss, since, at the limiting value, with $I = 0$, $(c = \infty)$, we obtain $M_{\max} = 2\,\mathrm{EI}\, \frac{\Delta f}{a^2}$. At the other limit (with an infinitely rigid truss), $I = \infty$, $(c = 0)$, $\eta = 0$, hence,

$$M_{\max} = \frac{1}{4} H' . \Delta f \text{ .... } (127^a.$$

The last expression gives pretty close results even for trusses of moderate stiffness.

The sag Δf at the crown, caused by an elongation of the cable amounting to $\Delta L = \omega . t . l$, is given with sufficient accuracy by

$$\Delta f = \left(\frac{a}{f} + \frac{1}{2}\frac{f}{a}\right)\frac{\Delta L}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (128.$$

The equations (127) and (127[a]) apply also in the case where the crown deflection Δf is caused by a uniform load covering the entire span, if H is the total horizontal tension resulting from this load.

Consequently it cannot be asserted that the introduction of a hinge in the stiffening truss will wholly eliminate the secondary stresses due to temperature variation.

6. Secondary Stresses. Up to this point in the theory of the stiffened suspension bridge it has been assumed that there is no friction either at the pins of the chain-cable or at the panel-points of the truss. The effect of riveted connections at the latter points is the same as ordinarily given in discussions of secondary stresses in framed structures;* so that we need to consider only the effect of friction at the joints of the chain upon the values of the horizontal tension and of the external forces acting on the stiffening truss. In wire-cable suspension bridges, the resistance of the stiffness of the cable takes the place of frictional resistance.

If d denotes the diameter of the pins, ϕ the coefficient of friction, T the cable tension, then the moment taken up by the joint-friction cannot exceed $M_\phi = \phi \,.\, T \,.\, \frac{d}{2}$ or approximately $M_\phi = \phi . H . \frac{d}{2}$; the moment transmitted to the stiffening truss will then be

$$M = \mathbf{M} - H.y - M_\phi = \mathbf{M} - H\left(y + \phi \cdot \frac{d}{2}\right).$$

The expressions for H established in § 5 (equs. 69 to 73) need not be altered, therefore, except to increase the arch ordinates y by $\phi \,.\, \frac{d}{2}$. The latter quantities, however, will always be very insignificant compared to y; even with the full effect of friction, assuming $\phi = 0.2$, the value of $\phi \frac{d}{2}$ will never exceed a few centimeters or millimeters; consequently this correction may be completely neglected in the expressions for H. Thus the value of the horizontal tension is not sensibly affected by friction in

* Handbuch des Brückenbaues, Chap. IX: Konstruktion der eisernen Balkenbrücken. Leipzig, 1906.

the chain or stiffness in the cable. The external forces acting on the stiffening truss likewise remain unaffected; at the most, the bending moments may be diminished by the friction-moment $\phi \cdot H \cdot \frac{d}{2}$.

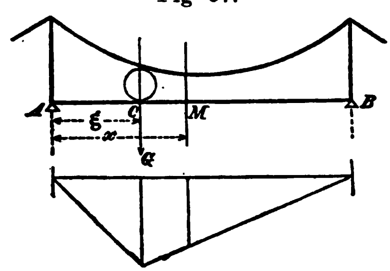
If the members of the chain are not pin-connected, but riveted instead, the stiffness of the chain must be added to that of the stiffening truss; and the moment $M = \mathbf{M} - Hy$ must be shared by both of these structural elements in the relative proportion of their moments of inertia. If I_0 is the moment of inertia of the cross-section of the chain, and I_1 that of the stiffening truss, then the chain will carry a bending moment of $M' = \frac{I_0}{I_0 + I_1} \cdot M$, and the truss will carry $M'' = M - M'$.

In pin-jointed chains, M' cannot exceed the value $\phi \cdot H \cdot \frac{d}{2}$. From the value of M' we may compute the resulting secondary fiber stresses in the chain.

§ 8. Computation of the Deflections.

1. Deflections Due to Loading. By Castigliano's Theorem, the deflection at any point of an elastic system may be obtained as the differential coefficient of the work of deformation with respect to a force supposed to be acting at the given point in the direction of the deflection.

Fig- 37.



On the truss $A\ B$, Fig. 37, let a concentration G be applied at a distance ξ from the abutment. Let the resulting axial forces and moments in the structure be denoted by N_ξ and M_ξ. Then the deflection at any section having the abscissa x will be,

$$\eta_x = \frac{dW}{dG_x} = \int \frac{N_\xi}{EA} \cdot \frac{dN_x}{dG_x} \cdot ds + \int \frac{M_\xi}{EI} \cdot \frac{dM_x}{dG_x} \cdot ds \quad \ldots\ldots (129.$$

where N_x and M_x denote the axial force and moment producible by a load G_x applied at x. Writing for the horizontal tensions corresponding to the load-positions ξ and x, $H_\xi = h_\xi . G$ and $H_x = h_x . G$, respectively, where the coefficients h_ξ and h_x, representing the horizontal tensions per unit load, may be obtained from the previously established equations (69) to (75ª) or from the H-curve and may therefore be considered as known quantities, there results

$$\frac{dN_x}{dG_x} = \frac{dN_x}{dH_x} \cdot h_x, \qquad \frac{dM_x}{dG_x} = \left(\frac{dM_x}{dG_x} - y \cdot h_x\right);$$

we thus obtain for a single-span suspension bridge without a central hinge:

Cable:
$$\frac{dW}{dG_x} = \sum_0^l H_\xi \cdot \frac{\lambda}{a} \cdot \frac{\lambda}{EA} \cdot h_x \cdot \frac{\lambda}{a}$$
$$= \frac{1}{EA_0} \cdot h_\xi \cdot h_x \cdot \sum_0^l \frac{\lambda^2}{a} \cdot G$$

Backstays:
$$\frac{dW}{dG_x} = \Sigma\, H_\xi \,.\, sec\, a_1 \cdot \frac{2l_1\, sec\, a_1}{EA_0\, sec\, a_1} \cdot h_x\, sec\, a_1$$
$$= \frac{2}{EA_0}\, h_\xi \,.\, h_x \,.\, l_1\, sec^2\, a_1 \,.\, G$$

Suspension Rods: $\frac{dW}{dG_x} = \Sigma H_\xi \cdot \frac{\Delta^2 y}{a} \cdot \frac{y'}{EA_2} \cdot h_x \cdot \frac{\Delta^2 y}{a}$

$$= \frac{1}{EA_2} \cdot h_\xi \cdot h_x \cdot \Sigma y' \left(\frac{\Delta^2 y}{a}\right)^2 \cdot G$$

Truss: $\frac{dW}{dG_x} = \frac{1}{EI} \int \left(\mathrm{M}_\xi - H_\xi \cdot y\right) \cdot \left(\frac{d\,\mathrm{M}_x}{dG_x} - y\,h_x\right) dx$

$$= \frac{1}{EI} \left[\mathrm{m}_x + h_\xi \cdot h_x \cdot \mu - h_x . m_\xi - h_\xi . m_x\right] . G$$

Here m_x denotes the moment at the point M of a simple beam produced by a loading composed of the ordinates of the moment-curve for a unit load at C. For this we obtain the expressions

$$\left.\begin{array}{l} \text{for } x < \xi, \ \mathrm{m}_x = [2l\xi - \xi^2 - x^2]\, \frac{x\,(l-\xi)}{6l} \\ \text{for } x > \xi, \ \mathrm{m}_x = [2lx - x^2 - \xi^2]\, \frac{\xi\,(l-x)}{6l} \end{array}\right\} \quad \ldots (130.$$

Furthermore, m_ξ and m_x represent the moments producible at the sections C and M, respectively, in a simple beam, when the loading consists of the area between the cable-curve and its chord. μ is the doubled static moment of the same area about the chord $\left(\mu = \int y^2\,dx\right)$. Since, by equ. (74), $h_\xi = \frac{m_\xi}{\mu + cp}$ and $h_x = \frac{m_x}{\mu + cp}$, therefore $m_\xi . h_x = m_x \cdot h_\xi$. Adding together the several values of $\frac{dW}{dG_x}$, we obtain

$$\left.\begin{array}{l} \eta_x = \left[\frac{h_x h_\xi}{EA_0}\left(l - \sum_0^l \frac{y}{a}\Delta^2 y + 2l_1 \sec^2 \alpha_1 + \frac{A_0}{A_2}\Sigma y'\left(\frac{\Delta^2 y}{a}\right)^2\right)\right. \\ \qquad \left. + \frac{1}{EI}\left(\mathrm{m}_x + h_x . h_\xi . \mu - 2\,h_\xi \cdot m_x\right)\right] . G \end{array}\right\} \ldots (131.$$

Noting that the first bracketed expression in the above equation, with the notation previously introduced, $= \frac{A_0}{I}\,cp$, and that $m_x = h_x\ (\mu + cp)$, we finally obtain the following simple formula for the deflection of the truss at M:

$$\eta_x = \frac{1}{EI}\left[\mathrm{m}_x - h_\xi \cdot m_x\right] . G \ldots\ldots\ldots\ldots\ldots\ldots (131^a$$

which could also have been derived directly from the differential equation of the elastic curve $\frac{d^2\eta}{dx^2} = -\frac{M}{EI}$. In the above expression for η, m_x is determined by equ. (130), h_ξ by equs. (71) to (75[a]), and m_x by the relation

$$m_x = \frac{l-x}{l}\int_0^x x\,y\,d\,x + \frac{x}{l}\int_x^l y\,(l-x)\,d\,x. \;..(132^{a}.$$

or, for a parabolic cable,

$$m_x = \frac{f\,x}{3\,l^2}(x^3 - 2\,l\,x^2 + l^3)\ldots\ldots\ldots\ldots\ldots (132^{a}.$$

The ordinates of the elastic curve may also be obtained by a graphic construction. This is essentially the same as the method used for the two-hinged arch in Figs. 1[a] to 1[f] (Plate I). Thus, if we write the expression for η in the form

$$\eta = \frac{h_\xi}{E\,I}\left(\frac{\mathrm{m}_x}{h_\xi} - m_x\right)\cdot G,$$

then that quantity may be represented as the difference between the ordinates of two funicular polygons, one of which (I) coincides with the H-curve, while the other (III) is constructed for an imaginary loading consisting of the simple moment-diagram (Fig. 1[d]). The latter is drawn by means of the force polygon (Fig. 1[e]) having H_ξ for its pole distance. If z_1 and z_3 are the ordinates of the two funicular polygons, measured to the scale of lengths, and if a is the width of the strips into which the load-diagrams are divided, and p is the pole distance of the force polygons (Figs. 1[b] and 1[f]) which were used for the construction of the funicular polygons I and III, then we have

$$\eta = \frac{a\,p}{E\,I}\,(z_3 - z_1)\cdot H_\xi$$

so that the scale for measuring the deflection ordinates is $\frac{EI}{H_\xi \cdot a \cdot p}$ times the scale of lengths.

In the example on page 37, $I = 0.0625$ m.^{4}, $a = 2.5$ m., $p = 35$ m., $H_\xi = \frac{11.45}{15.0}\,G = 0.764\,G$, $E = 20{,}000{,}000$ t./sq. m.; if the scale of lengths adopted is 1 m. = 2 mm., then the scale for the platted deflection ordinates will be 1 mm. (actual) = 37.4 mm. on the drawing.

With the aid of the deflection diagram drawn for a unit load at C, it becomes a simple matter to determine the deflection producible at C by any system of concentrations or by any continuous loading. By the principle of the reciprocity of deflections (Maxwell's Theorem), the deflection at C caused by any load at M is equal to the deflection producible at M by an equal load at C. Consequently, the curve of deflections (η) for a unit load at C is at the same time an influence line for deflections at the point C; hence to find the total deflection at

C, multiply each load by the corresponding ordinate of the deflection curve, and add the products thus obtained.

The above construction, based on the assumption of a constant moment of inertia of the stiffening truss, may also be extended to the case of a variable moment of inertia. If I is the moment of inertia at any section and I_0 a fixed moment of inertia, then it is simply necessary to reduce the ordinates of the load-diagrams, i. e., the area between the cable-curve and its chord and the area of the simple moment diagram, in the ratio $I_0 : I$, and proceed with these reduced areas exactly as in the above construction. In the example of the arch in Plate I, the variation of the moments of inertia was thus taken into account.

For any specified loading, the deflections of the stiffening truss may be computed in another manner. On the truss there are acting, in addition to the external loads which produce the moments $\mathbf{M}$, the upward-directed suspender-forces, $H\frac{\Delta^2 y}{a}$, or $H\frac{d^2 y}{dx^2}$ per unit length. The former loads produce a deflection at a distance x of

$$\left.\begin{aligned} \eta' &= \frac{l-x}{l}\int_0^x \frac{\mathbf{M}}{EI}x\,dx \quad + \frac{x}{l}\int^l \frac{\mathbf{M}}{EI}(l-x)\,dx \\ &= \frac{1}{EI}\,(G\,\mathrm{m}_x) \end{aligned}\right\} \qquad (133.$$

while the latter loads, the suspender-forces, in the case of a parabolic cable, are uniformly distributed and $= \frac{8f}{l^2}\cdot H$; they produce a deflection $-\eta''$ which, with a constant I in the stiffening truss, is determined by the equation

$$\eta'' = \frac{1}{3EI}x\,(l^3 - 2lx^2 + x^3)\,\frac{f}{l^2}\cdot H = \frac{1}{EI}\cdot H\cdot m_x \qquad (134.$$

The resulting deflection of the truss is therefore $\eta = \eta' - \eta''$, which coincides with equ. (131ª). We thus find the deflection at the mid-point produced by a uniform load of p per unit length covering the entire span to be given by the equation

$$\eta = \frac{5}{384}\left(1 - \frac{8}{5N}\right)\frac{pl^4}{EI} \qquad (135.$$

where N denotes the denominator of equ. (75) or (75ª).

If merely the half-span is loaded with p per unit length, then the deflection at the quarter point will be, by (133) and (134),

$$\left.\begin{array}{l} \text{in the loaded half,} \quad \eta = \dfrac{1}{6144}\left(31 - \dfrac{57}{2}\cdot\dfrac{8}{5N}\right)\dfrac{p\,l^4}{EI} \\ \text{in the unloaded half,} \quad \eta = -\dfrac{1}{6144}\left(\dfrac{57}{2}\cdot\dfrac{8}{5N} - 26\right)\dfrac{p\,l^4}{EI} \end{array}\right\} \quad \ldots (136.$$

By equ. (75), N will always be greater than $\frac{8}{5}$; substituting this minimum value in (136) we obtain the upward or downward deflections at the quarter points:

$$\eta = \frac{1}{2}\cdot\frac{5}{384}\frac{p}{EI}\left(\frac{l}{2}\right)^4.$$

For the *stiffening truss with central hinge,* we must substitute in the deflection formula, equ. (131), the values $h_\xi = \frac{\xi}{2f}$ and $h_x = \frac{x}{2f}$, where it is presupposed that ξ and $x < \frac{l}{2}$. Denoting the first bracketed expression, which represents the effect of the axial forces, by the symbol Σ, equ. (131) takes the form:

$$\left.\begin{array}{l} \eta_x = \dfrac{G\,x\,\xi}{4\,f^2 E\,A_0}\Sigma \\ \quad + \dfrac{G}{EI}\left[\mathbf{m}_x + \dfrac{x\,.\,\xi}{4\,f^2}\mu - \dfrac{1}{2f}\left(m_x\,\xi + m_\xi\,x\right)\right] \end{array}\right\} \quad \ldots\ldots (137.$$

The symbols $\mathbf{m}_x$, μ, m_x and m_ξ have the significations previously specified. For $x > \frac{l}{2}$,

$$\left.\begin{array}{l} \eta_x = \dfrac{G\,(l-x)\,\xi}{4f^2 EA_0}\Sigma + \dfrac{G}{EI}\Big[\mathbf{m}_x + \dfrac{(l-x)\,\xi}{4f^2}\mu - \dfrac{1}{2f}\left[m_x\xi\right. \\ \qquad\qquad\qquad\qquad \left.+ m_\xi\ (l-x)\right]\Big] \end{array}\right\} \quad \ldots\ldots (138.$$

Substituting the values of m_x, μ and m_ξ which obtain for a parabolic cable, there result the following expressions:

For $x < \xi < \frac{l}{2}$:

$$\left.\begin{array}{l} \eta_x = \dfrac{Gx\xi\Sigma}{4f^2 EA_0} + \dfrac{G\,x}{30\,EI\,l^2}\left[4\,l^3\,\xi - 15\,\xi^2 l^2 + 15\,\xi l\,(x^2+\xi^2)\right. \\ \qquad\qquad \left. -5\,\xi\,(x^3+\xi^3) - 5\,x^2 l^2\right] \end{array}\right\} \quad (139.$$

For $\xi < x < \frac{l}{2}$:

$$\left.\begin{array}{l} \eta_x = \dfrac{G\,x\,\xi\Sigma}{4f^2 EA_0} + \dfrac{G\xi}{30EIl^2}\left[4\,l^3\,x - 15\,x^2 l^2 + 15\,x\,l\,(x^2+\xi^2)\right. \\ \qquad\qquad \left. -5x\,(x^3+\xi^3) - 5\xi^2 l^2\right] \end{array}\right\} \quad \ldots\ldots (140.$$

For $x > \frac{l}{2} > \xi$:

$$\eta_x = \frac{G\xi(l-x)\Sigma}{4f^2EA_0} + \frac{G\xi(l-x)}{30EIl^2}[5\xi^2(l-\xi)+5x(l-x)^2-l^3] \quad ..(141.$$

Here, again, we may apply the graphic method for determining the deflection-ordinates. Thus, equ. (137) may be written

$$\eta_x = \frac{G}{EI}(m_x - h_\xi m_x) + \delta\frac{2x}{l} \quad \ldots\ldots\ldots\ldots (142.$$

where

$$\delta = \left[h_\xi\left(\mu + \frac{I_0}{A_0}\Sigma\right) - m_\xi\right]\frac{l}{4f}\cdot\frac{G}{EI} \quad \ldots\ldots (143.$$

The bracketed expression in equ. (142) may be represented, as before, by the intercepts between the two funicular polygons I and III. (Figs. 1^a to 1^f, Pl. I), where the simple moment diagram (Fig 1^d) is obtained from the force polygon (Fig. 1^e) constructed with the pole distance $H_\xi = G \cdot \frac{\xi}{2f}$. To obtain the actual deflections, the above intercepts must be increased by the ordinates of a triangle whose altitude, at the center of the span, is $\delta \cdot \frac{EI}{H_\xi \cdot ap}$.

If, as before, η' and η'' denote the deflections specified by equs. (133) and (134) for the hingeless structure, produced by the external loads and by the suspender-forces respectively; if H is the horizontal tension produced in the hinged structure by the given loading and H_0 is the horizontal tension producible in an identical structure *without* the hinge; and if N is the denominator of the expression for H_0 in equ. (75), then by a transformation of equ. (137) we obtain,

$$\eta_x = \eta' - \eta'' + \frac{N}{12EI}fl^2(H-H_0)\frac{2x}{l} \quad \ldots\ldots (144.$$

From this we find the deflection at the mid-point produced by a uniform full-span load of p per unit length to be

$$\eta = \frac{1}{96}\left(N - \frac{8}{5}\right)\frac{pl^4}{EI} \quad \ldots\ldots\ldots\ldots\ldots\ldots (145.$$

and the deflections at the quarter-points produced by a half-span load are, in the loaded half,

$$\eta = \frac{1}{2}\cdot\frac{5}{384}\cdot\frac{p}{EI}\left(\frac{l}{2}\right)^4 + \frac{1}{384}\left(N - \frac{8}{5}\right)\frac{pl^4}{EI}$$

in the unloaded half, $\ldots\ldots$ (146.

$$\eta = -\frac{1}{2}\cdot\frac{5}{384}\cdot\frac{p}{EI}\left(\frac{l}{2}\right)^4 + \frac{1}{384}\left(N - \frac{8}{5}\right)\frac{pl^4}{EI}$$

2. Deflections Produced by Temperature Variations or by Displacements of the Cable Supports. Equ. (131^a), which defines the deflection-ordinate at a distance x from the end of the span, applies also to the case of deflections produced not by the loading but by temperature effects or by a yielding of the towers or backstays. In this case, $\mathbf{m}_x = 0$; and $G.h$ must be equal to the horizontal tension H_t caused by the given influence. Consequently,

$$\eta = -\frac{1}{EI} H_t . m_x \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (147.$$

Here H_t must be determined by equs. (119) to (123) or, if the deflection is caused by a yielding of the cable supports, by equ. (122), while m_x is given by equ. (132) or (132^a). Since the quantity m_x is proportional to the horizontal tension producible by a concentration applied at the point x, the deflections of the stiffening truss are given by the ordinates of the H-curve, i. e., the funicular polygon I (Fig. 1^a, Pl. I), where, with the previous notation, the scale unit to be adopted is $\frac{EI}{H_t\, a\, p}$ times that of the scale of lengths.

In the stiffening truss interrupted by a central hinge, any stretch in the cable will cause a sag at the point $x \left(< \frac{l}{2}\right)$; if the bending of the truss is neglected, this sag will amount to

$$\eta = \Delta f \cdot \frac{2x}{l},$$

where Δf is determined by equs. (39) to (44).

§ 9. The More Exact Theory for the Stiffened Suspension Bridge.

The theory developed in the preceding paragraphs (§§ 5-8) gives satisfactory results only for those systems in which the elastic deformations are limited in amount. In the systems treated above, the admissibility of that approximate method of design depends upon the degree of thoroughness with which the stiffening truss performs its function of limiting the mobility of the system. If the truss is provided with but small chord-sections or has but small depth, that is, if it has a small moment of inertia and consequently great flexibility, or if it is provided with an intermediate hinge, then, under certain cases of loading, the deformation of the system ought no longer to be neglected in determining the stresses.

1. Truss without a Middle Hinge. Single Span. Let H_0 denote the horizontal tension in the cable in its initial position of equilibrium (when it is carrying no live load), in which condition it is assumed to form a parabolic curve. The stiffening truss is then without stress. Let the suspender-forces, under this loading, be denoted by s_0 (per unit length), and the dead weight of the cable by k; then by equ. (51) we obtain the following differential equation of the equilibrium curve:

$$H_0 \frac{d^2y}{dx^2} = -(s_0 + k) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\alpha.$$

On bringing a live load upon the structure, of intensity q per unit length, let the suspender-forces change to s_1, the horizontal tension to $(H_0 + H)$, the ordinates of the equilibrium curve to $(y + \eta)$; we may again write the relation between these quantities:

$$-(H_0 + H) \frac{d^2(y+\eta)}{dx^2} = s_1 + k \quad \ldots\ldots\ldots\ldots (\beta.$$

By the addition of the live load q there is also produced a deflection of the stiffening truss. Neglecting the elongation of the suspension rods, the deflection of the truss at any section may be equated to the sag (η) of the cable at the same section; so that η is the ordinate of the elastic curve of the truss for which we may write the differential equation

$$EI \frac{d^4\eta}{dx^4} = q - (s_1 - s_0) \quad \ldots\ldots\ldots\ldots\ldots\ldots (\gamma.$$

The addition of equs. (α), (β) and (γ) yields

$$EI\frac{d^4\eta}{dx^4}-(H_0+H)\frac{d^2\eta}{dx^2}-H\frac{d^2y}{dx^2}=q.$$

Introducing the abbreviation,

$$\frac{H+H_0}{EI}=c^2 \qquad (148$$

and assuming that I and q are constant within the limits of integration, the integration of the above differential equation yields

$$\frac{d^2\eta}{dx^2}=Ae^{cx}+Be^{-cx}-\frac{H}{H+H_0}\left(\frac{q}{H}+\frac{d^2y}{dx^2}+\frac{1}{c^2}\frac{d^4y}{dx^4}+..\right);$$

or since, on account of the initial parabolic form of the cable, $\frac{d^2y}{dx^2}=-\frac{8f}{l^2}$ and $\frac{d^4y}{dx^4}=0$,

$$\frac{d^2\eta}{dx^2}=Ae^{cx}+Be^{-cx}-\frac{H}{H+H_0}\left(\frac{q}{H}-\frac{8f}{l^2}\right) \qquad (\delta.$$

The forces $q-(s_1-s_0)$ acting on the truss produce a bending moment $M=\mathbf{M}-(H+H_0)(y+\eta)+H_0\,.\,y$, i. e.,

$$M=\mathbf{M}-(H+H_0)\eta-H\,.\,y \qquad (149.$$

where $\mathbf{M}$ denotes the moment producible by the loading q in a simply supported beam. Consequently,

$$\frac{d^2\eta}{dx^2}=-\frac{M}{EI}=+c^2\eta-\frac{c^2}{H+H_0}(\mathbf{M}-Hy) \qquad (\epsilon.$$

Equating the expressions (δ) and (ϵ), substituting $y=\frac{4f}{l^2}x\,.\,(l-x)$ and replacing the arbitrary constants A and B by $C_1=\frac{AEI}{H}$ and $C_2=\frac{BEI}{H}$, we finally obtain

$$\left.\begin{array}{l}\eta=\frac{H}{H+H_0}\\ \cdot\left(C_1e^{cx}+C_2e^{-cx}+\frac{8f}{c^2l^2}-\frac{4f}{l^2}x(l-x)+\frac{\mathbf{M}}{H}-\frac{q}{Hc^2}\right)\end{array}\right\} \qquad (150.$$

The arbitrary constants C_1 and C_2 are determined by the condition that at $x=0$ and $x=l$, we must have $\eta=0$. If the load q covers but a fraction of the span, different values must be given to the constants in the equations of the elastic curve for the loaded and the unloaded segments. The necessary equations are then given by the condition that, at the division point, the two adjacent segments must have identical values of η and $\frac{d\eta}{dx}$.

Substituting the value of η in equ. (149), we obtain an expression for the moment

$$M = -H\left[C_1 e^{cx} + C_2 e^{-cx} + \frac{1}{c^2}\left(\frac{8f}{l^2} - \frac{q}{H}\right)\right] \quad \ldots (151.$$

From equ. (149) we observe that the moment M is no longer proportional to the intensity of the load q which produces it, but depends in amount also upon the horizontal tension H_0 which originally existed in the structure before the application of the load q. In distinction from the approximate theory, the term $-(H + H_0).\eta$ represents the effects of the deformation and the initial tension H_0 of the cable. This correction is in a desirable direction in the case of the stiffened suspension bridge, since, for the loading which produces a maximum positive moment, η is positive, so that the bending moment will be diminished; and the same is true with respect to the numerical value of the negative moment. In the arch construction, however, the reverse obtains, for there the deflections of the structure tend to increase the bending moments. The ratio of the missing term $-(H + H_0)\,\eta$ to the approximate value of the moment $(\mathbf{M} - Hy)$ may, under certain conditions, become very large even with a rigid stiffening truss, particularly at those sections where $(\mathbf{M} - H\,y)$ is very small or equal to zero. On the other hand, the effect of this deformation will not be sensible for those cases of loading which produce maximum bending moments unless the coefficient c is very large, corresponding to a very small value of I. (Compare the examples calculated below.)

The bending moments produced in the stiffening truss by any given loading may be determined exactly by equ. (151), if the value of the horizontal tension H is known. Since the deformations of the cable will in general be but very small, we may state in anticipation that these values of H will differ but slightly from the values given by the foregoing approximate equations (§§ 5 and 6). It will therefore usually suffice to use the approximate values of H in computing the bending moments by equ. (151).

To determine H more accurately, we make use of the principle that, for small deformations, the work done by the forces $(s_1 + k)$ acting on the cable must equal the work of deformation of the internal stresses in the cable; hence, with the previously adopted notation,

$$\int_0^l (s_1 + k)\,.\,\eta\,.\,dx = \frac{(H + H_0)\,H}{E\,A_0}\left(\Sigma\,\frac{\lambda^3}{a} + 2\,l_1\,sec^2\,a_1\right)$$

From equ. (β) we may write, approximately,

$$(s_1+k)=-(H+H_0)\frac{d^2y}{dx^2}=\frac{8f}{l^2}(H+H_0);$$

also, adopting the abbreviation $\Sigma\frac{\lambda^2}{a}+2l_1\sec^2\alpha_1=L$, so that, for a parabolic cable,

$$L=l\left(1+\frac{16}{3}\frac{f^2}{l^2}\right)+2l_1\sec^2\alpha_1,$$

and substituting the value of η from equ. (150), the above equation of virtual work becomes

$$\int_0^l\left[C_1e^{cx}+C_2e^{-cx}+\frac{1}{H}\left(\mathrm{M}-\frac{q}{c^2}\right)+\frac{8f}{l^2c^2}-\frac{4fx(l-x)}{l^2}\right]dx$$
$$=c^2\frac{I}{A_0}\frac{l^2L}{8f}.$$

Solving this equation for the horizontal tension, we obtain

$$H=\frac{\int_0\left(\mathrm{M}-\frac{q}{c^2}\right)dx}{-\int_0^l(C_1e^{cx}+C_2e^{-cx})\,dx+\frac{2}{3}fl-\frac{8f}{lc^2}+c^2\frac{I}{A_0}\frac{l^2L}{8f}}\ \ldots\ (152$$

In order to apply this formula, it is necessary first to find the approximate value of H by the preceding method, namely by the equation $H=\dfrac{\int_0^l \mathrm{M}y\,dx}{\int y^2\,dx+\frac{IL}{A}}$, and to use this value in computing the quantities c, C_1 and C_2 appearing in the above expression.

Equations (151) and (152) hold good for any condition of loading. Let us first apply them to the case of a single concentration (P) located at the distances ξ and $l-\xi=\xi'$ from the respective abutments. For this loading, the following values are obtained for the arbitrary constants:

For the segment from 0 to ξ:

$$C_1=\frac{1}{e^{cl}-e^{-cl}}\left\{\frac{P}{2cH}(e^{-c\xi'}-e^{c\xi'})-\frac{8f}{c^2l^2}\cdot(1-e^{-cl})\right\}$$
$$C_2=\frac{1}{e^{cl}-e^{-cl}}\left\{\frac{P}{2cH}(e^{c\xi'}-e^{-c\xi'})+\frac{8f}{c^2l^2}(1-e^{cl})\right\}$$

For the segment from ξ to l: ....(153.

$$C_3=\frac{1}{e^{cl}-e^{-cl}}\left\{\frac{P}{2cH}(e^{-c\xi}-e^{c\xi})-\frac{8f}{c^2l^2}(1-e^{-cl})\right\}$$
$$C_4=\frac{1}{e^{cl}-e^{-cl}}\left\{\frac{P}{2cH}(e^{c\xi}-e^{-c\xi})+\frac{8f}{c^2l^2}(1-e^{cl})\right\}$$

We thus obtain:

$$H=\frac{\frac{1}{2}P\xi\xi'}{\frac{1}{c^2}\left[K-\frac{8f}{l}\right]+\frac{2}{3}fl+c^2\frac{I}{A_0}\cdot\frac{l^2L}{8f}} \quad \ldots\ldots (154.$$

where, for abbreviation,

$$K=\frac{2}{e^{cl}-e^{-cl}}\left\{\frac{P}{2H}\left[(e^{cl}-e^{-cl})-(e^{c\xi}-e^{-c\xi})-(e^{c\xi'}-e^{-c\xi'})\right]+\frac{8f}{cl^2}(e^{cl}+e^{-cl}-2)\right\} \quad (154^a.$$

The bending moments in the stiffening truss are then to be computed by equ. (151).

With this more exact theory, as previously observed, there is no longer any proportionality between the applied loads and the internal stresses. The method of influence lines is therefore inapplicable.

If the loading consists of a load $g'l$ uniformly distributed over the entire span and a similar load $p\lambda$ covering a distance λ from the left abutment, then the arbitrary constants of equ. (150) are defined by the following expressions:

For the segment from $x=0$ to $x=\lambda$

$$Hc^2C_1=\frac{p}{2}\,\frac{2-e^{c\lambda}-e^{c(2l-\lambda)}}{1-e^{2cl}}-\left(\frac{8f}{l^2}\cdot H-g'\right)\frac{1}{1+e^{cl}}$$

$$Hc^2C_2=-Hc^2C_1-\left(\frac{8f}{l^2}\ H-p+g'\right) \quad \ldots (155.$$

For the segment from $x=\lambda$ to $x=l$,

$$Hc^2C_3=Hc^2C_1-\frac{1}{2}p\,e^{-c\lambda}$$

$$Hc^2C_4=Hc^2C_2-\frac{1}{2}p\,e^{c\lambda}$$

and equ. (151) yields the following expressions for the moments:

For $x=0$ to $x=\lambda$,

$$M=-\frac{1}{c^2}\left[Hc^2C_1e^{cx}+Hc^2C_2e^{-cx}+\frac{8f}{l^2}H-(p+g')\right] \quad . (156.$$

For $x=\lambda$ to $x=l$,

$$M=-\frac{1}{c^2}\left[Hc^2C_3e^{cx}+Hc^2C_4e^{-cx}+\frac{8f}{l^2}H-g'\right]$$

Here H is the horizontal tension produced by the loading $g'l+p\lambda$, $c^2=\frac{H+H_0}{EI}$, and H_0 denotes the horizontal tension corresponding to the initial parabolic curve of the cable and the

unstressed condition of the stiffening truss. In determining the value of c, the value of H given by the approximate theory may be used.

If only the uniform load g', covering the entire span, is applied, the constants reduce to

$$\left.\begin{aligned} C_1 &= \frac{1}{Hc^2}\left(g' - \frac{8f}{l^2}H\right)\frac{1}{1+e^{cl}}\,; \\ C_2 &= \frac{1}{Hc^2}\left(g' - \frac{8f}{l^2}H\right)\frac{e^{cl}}{1+e^{cl}}\,; \end{aligned}\right\} \ldots\ldots (157.$$

and the moment becomes

$$M = \frac{1}{c^2}\left(g' - \frac{8f}{l^2}H\right)\left[1 - \frac{e^{cx} + e^{c\,(l-x)}}{1+e^{cl}}\right] \ldots\ldots (158.$$

Example 1. Double-Track Railway Bridge. Span l = 150 meters. Steel-wire cable: Rise = 20 m., horizontal length of back-stays l_1 = 75 m., $\tan\alpha_1 = 0.4$, cable-section A_0 = 328 cm². Stiffening truss: parallel-chords, depth = 7 m., mean moment of inertia per truss I = 69,473,000 cm⁴. = 0.69473 m⁴. Dead load per truss g = 2.4 tonnes, live load p = 4.0 tonnes per linear meter. It is assumed that the total dead load is carried by the cable alone, and hence there are no bending moments in the stiffening truss when the bridge is unloaded. To attain this condition, the stiffening truss is provided with a central hinge during erection and not until the construction is completed and the suspension rods adjusted is this hinge to be replaced by a rigid connection. We have

$$L = l\left(1 + \frac{16}{3}\frac{f^2}{l^2} + \frac{2\,l_1\sec^2\alpha_1}{l}\right) = 2.255\,l.$$

Hence the denominator of equ. (75ᵃ) becomes

$$N = \frac{8}{5} + \frac{3 \times 0.69473}{0.0328 \times 400} \times 2.255 = 1.9582.$$

By the above assumption,

$$H_0 = \frac{1}{8} \times \frac{\overline{150}^2}{20} \times 2.4 = 337.5 \text{ tonnes.}$$

With the half-span loaded, by the approximate theory (equ. 82),

$$H = \frac{l^2}{5\,Nf}\cdot\frac{p}{2} = 114.90 \times \frac{4}{2} = 229.8\,\text{t.}$$

With the same half-load, the bending moments computed by the *approximate theory* are as follows: ($M = \mathbf{M} - Hy$.)

$$\begin{cases} \text{At } x = \frac{1}{4}\,l, & M_1 = \frac{1}{16}\,pl^2 - Hy_1 = 5625 - 229.8 \times 15 = 2178 \text{ t. m.} \\ \text{At } x = \frac{1}{2}\,l, & M_2 = \frac{1}{16}\,pl^2 - H\,.\,f = 5625 - 229.8 \times 20 = 1029 \text{ t. m.} \\ \text{At } x = \frac{3}{4}\,l, & M_3 = \frac{1}{32}\,pl^2 - Hy_1 = 2812.5 - 229.8 \times 15 = -634.5 \text{ t. m.} \end{cases}$$

In applying the *more exact theory*, we have:

$$c^2 = \frac{H_0 + H}{EI} = \frac{567.3}{20{,}000{,}000 \times 0.69473} = 0.000040835$$

$$\frac{1}{c^2} = 24{,}488.8 \qquad c = 0.00639 \qquad cl = 0.9585 \qquad \frac{8f}{l^2} \cdot H = 1.6348$$

Using these values and putting $g' = 0$, $\lambda = \frac{1}{2} l$, in equs. (155) and (156), we obtain the bending moments:

$$\begin{cases} \text{At } x = \frac{1}{4} l, & M_1 = \frac{1}{c^2} \times 0.0847565 = 2075.6 \text{ t. m.} \\ \text{At } x = \frac{1}{2} l, & M_2 = \frac{1}{c^2} \times 0.038268 = 937.1 \text{ t. m.} \\ \text{At } x = \frac{3}{4} l, & M_3 = -\frac{1}{c^2} \times 0.027238 = -667.0 \text{ t. m.} \end{cases}$$

Comparing the two methods, it is seen that the exact design, which allows for the deformation of the cable under load, yields results for the positive moments M_1 and M_2 smaller by 4.7 and 8.8% respectively and for the negative moment M_3 larger by 4.9% than the corresponding results of the approximate theory. If the stiffening truss is more flexible, as in the following example, larger differences will appear.

If the entire span is loaded with $p = 4.0$ tonnes per linear meter, we find by the approximate theory,

$$H = 114.9 \times 4 = 459.6 \text{ t.}$$

and the moment at the mid-point

$$M = \frac{1}{8} pl^2 - Hf = 11250 - 459.6 \times 20 = 2058.0 \text{ t. m.}$$

The more exact method yields:

$$c^2 = \frac{797.1}{20{,}000{,}000 \times 0.69473} = 0.000050173$$

$$\frac{1}{c^2} = 19{,}931.0 \qquad cl = 1.06249$$

and by equ. (158), substituting for g' the value $p = 4.0$ t.,

$$\text{at } x = \frac{1}{2} l, \qquad M = 0.92195 \frac{1}{c^2} = 1837.6 \text{ t. m.}$$

This is about 10% less than the result of the approximate design.

Example 2. Highway Bridge. Span and rise as above. Stiffening truss, 2.5 meters high; mean moment of inertia $I = 7{,}812{,}000$ cm^4 $= 0.07812$ m^4. Cable-section $A_0 = 200$ cm^2. Dead load per truss, $g = 2.40$ tonnes, live load $p = 1.4$ t. per linear meter. It is again assumed that the total dead weight of the bridge is carried by the cable. With $L = 2.255\, l$, we find

$$N = \frac{8}{5} + \frac{3 \times 0.07812}{0.02 \times 400} \times 2.255 = 1.66606,$$

$$H_0 = \frac{1}{8} \times \frac{(150)^2}{20} \times 2.4 = 337.5 \text{ t.},$$

and, with the half-span loaded, by equ. (82),

$$H = 135.048 \times \frac{1.4}{2} = 94.53 \text{ t.}$$

For this half-load, the *approximate theory* yields the following values of the moments:

$$\begin{cases} \text{At } x = \frac{1}{4}l, & M_1 = 1968.75 - 94.53 \times 15 = 550.8 \text{ t. m.} \\ \text{At } x = \frac{1}{2}l, & M_2 = 1968.75 - 94.53 \times 20 = 78.15 \text{ t. m.} \\ \text{At } x = \frac{3}{4}l, & M_3 = 984.37 - 94.53 \times 15 = -433.57 \text{ t. m.} \end{cases}$$

Applying the *exact theory,* we have:

$$c^2 = \frac{H_0 + H}{E\,I} = \frac{432.05}{20{,}000{,}000 \times .07812} = .0002766,$$

$$\frac{1}{c^2} = 3{,}615.4, \quad c\,l = 2.49466, \quad \frac{8f}{l^2}H = 0.672213.$$

Using these values and $\lambda = \frac{1}{2}l$ in equ. (156), we compute the following moments:

$$\begin{cases} \text{At } x = \frac{1}{4}l, & M_1 = \frac{1}{c^2}\,0.12717 = 459.76 \text{ t. m.} \\ \text{At } x = \frac{1}{2}l, & M_2 = \frac{1}{c^2}\,0.01304 = 47.15 \text{ t. m.} \\ \text{At } x = \frac{3}{4}l, & M_3 = -\frac{1}{c^2}\,0.10701 = -386.89 \text{ t. m.} \end{cases}$$

The deviation from the approximate values amounts to about 17% in M_1, and about 40% of the approximate value in the small bending moment at the mid-point; and in every case the approximate theory makes the bending moments too large.

With the entire span loaded with $p = 1.4$ t. per linear meter,

$$H = 135.048 \times 1.4 = 189.06 \text{ t.}$$

and the moment at the mid-point is given by the approximate method as

$$M = \frac{1}{8}pl^2 - H\,.\,f = 3937.5 - 3781.2 = 156.3 \text{ t. m.}$$

For the more exact design we have:

$$c^2 = \frac{526.56}{1{,}562{,}400} = 0.0003370, \quad \frac{1}{c^2} = 2{,}967.2, \quad c\,l = 2.7537$$

and by equ. (158), for $x = \frac{l}{2}$, we obtain

$$M = 0.029203 \cdot \frac{1}{c^2} = 86.6 \text{ t. m.}$$

Hence the approximate design in this case yields a value too great by more than 40%.

2. Truss with a Middle Hinge. Let the stiffening truss have a constant moment of inertia I; and, in the unloaded condition, with only the dead load of g units per linear meter acting, at a definite temperature, let the form of the cable coincide with the equilibrium curve of the dead load, so that there is then no stress in the stiffening truss. Let H_0 be the horizontal tension of the cable when this condition obtains.

Upon applying a live load p, which increases the horizontal tension by H, or for any other cause, let the cable sag at the crown by an amount Δf, producing a bending in the stiffening truss. Writing for the half-span $\frac{l}{2}=a$, and denoting by η the deflection due to bending of the stiffening truss at a distance x from the abutment, then the total deflection at that point will be $\eta+\frac{x}{a}\,.\,\Delta f$, and the same amount will represent the increase in the ordinate y of the cable-curve if the stretching of the suspenders is neglected. The horizontal forces (H_0+H) corresponding to the initial position of the cable-crown will now change to $(H'_0+H')=\frac{f}{f+\Delta f}(H_0+H)$. If **M** denotes the moment of the loads $g+p$ in a simply supported beam, then the bending moment in the stiffening truss will be

$$M=\mathbf{M}-(H'_0+H')\left(y+\frac{x}{a}\Delta f+\eta\right)\ldots\ldots\ldots(\alpha.$$

and the differential equation of the elastic curve of the structure will be

$$\frac{d^2\eta}{dx^2}=c^2\eta-c^2\left(\frac{\mathbf{M}}{H'_0+H'}-y-\frac{x}{a}\Delta f\right)\ldots\ldots(\beta.$$

when, for abbreviation, we put

$$c^2=\frac{H'_0+H'}{EI}\doteq\frac{H_0+H}{EI}.$$

If we assume Δf independent of η, which is very nearly the case, then, for a parabolic initial form of the cable and for the cases of loading occurring in practice, the bracketed quantity in (β) will be a homogeneous algebraic expression of the second degree; let this expression be denoted by $F(x)$ and its second derivative with respect to x by $F''(x)$. Then the integration of (β) yields

$$\eta=Ae^{cx}+Be^{-cx}+F(x)+\frac{1}{c^2}F''(x)\quad\ldots\ldots\ldots(\gamma.$$

The constants A and B are given by the conditions that at $x=0$ and $x=a$, $\eta=0$. Substituting the expression (γ) in equation (α) gives the bending moment

$$M = -(H'_0 + H')(A e^{cx} + B e^{-cx}) - EI \cdot F''(x) \quad \ldots (\delta.$$

The design for any given loading now becomes a simple matter. With a full-span load of p per unit length we obtain

$$F(x) = \frac{x(a-x)}{a^2} \cdot \Delta f, \qquad c^2 = \frac{I}{EI} \cdot \frac{(g+p)a^2}{2f}$$

and the maximum moment at the quarter point

$$M_{\max} = \frac{(e^{\frac{1}{2}ca} - 1)^2}{e^{ca} + 1} 2EI \cdot \frac{\Delta f}{a^2} \quad \ldots\ldots\ldots\ldots\ldots (159.$$

When the half-span is loaded with p per unit length,

$$c^2 = \frac{1}{EI} \frac{(2g+p)a^2}{4f}$$

and the maximum moments at the quarter-points in the loaded and unloaded segments will be

$$\left.\begin{aligned} M_{1\max} &= \frac{(e^{\frac{1}{2}ca} - 1)^2}{e^{ca} + 1} \cdot \frac{2EI}{a^2} \cdot \frac{2(g+p)\Delta f + pf}{2g+p} \\ M_{2\max} &= \frac{(e^{\frac{1}{2}ca} - 1)^2}{e^{ca} + 1} \cdot \frac{2EI}{a^2} \cdot \frac{2g\Delta f - pf}{2g+p} \end{aligned}\right\} \ldots\ldots (160.$$

For a very stiff truss ($I = \infty$, $c = 0$), we have

$$\frac{(e^{\frac{1}{2}ca} - 1)^2}{e^{ca} + 1} \cdot \frac{2EI}{a^2} = \frac{1}{4}(H'_0 + H'),$$

so that the extra moment at the quarter points due to the sag Δf at the crown will be given with sufficient accuracy for most cases by the formula

$$M = \frac{1}{4}(H_0 + H) \cdot \Delta f \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (161.$$

The crown sag Δf, due to an elongation of the cable amounting to ΔL, is expressed with sufficient accuracy by the equation (128),

$$\Delta f = \left(\frac{a}{f} + \frac{1}{2}\frac{f}{a}\right) \cdot \frac{\Delta L}{2}$$

Example. If the bridge of the above example 1. is constructed with a middle hinge, then, with $p = 4.0$ t., $H + H_0 = \frac{1}{8} \frac{6.4\ (150)^2}{20} = 900$ t., $c^2 = \frac{H_0 + H}{EI} = 0.0000648$, $ca = 0.6037$; hence, by equ. (159), $M_{\max} = 216.83\ \Delta f$ or approximately, by equ. (161), $M_{\max} = \frac{900}{4} \cdot \Delta f = 225 \cdot \Delta f$. The elastic stretch of the cable will be-

$$\Delta L = \frac{H}{EA} \cdot L = \frac{562.5 \times 338}{2000 \times 328} = 0.290 \text{ m.},$$

hence

$$\Delta f = \left(\frac{75}{20} + \frac{1}{2} \cdot \frac{20}{75}\right) 0.145 = 0.563 \text{ m.}$$

and

$$M_{max} = 216.83 \times 0.563 = 122.1 \text{ t. m.}$$

With only the half-span loaded, $H + H_0 = \frac{1}{16} \frac{8.8 \ (150)^2}{20} = 618.75$ t., $c^2 = 0.0000445$, $c\,a = 0.5002$, and we obtain, by equ. (160),

$$M_{1\ max} = (1369.3 + 219.1\ \Delta f) \text{ t. m.}$$

$$M_{2\ max} = (-1369.3 + 82.2\ \Delta f) \text{ t. m.}$$

The approximate formula for the rigid truss gives,

$$M_{1,2\ max} = \pm \frac{1}{64}\, pl^2 + \frac{1}{4}\,(H_0 + H) \cdot \Delta f = (\pm 1406 + 154.7\, \Delta f) \text{ t. m}$$

Here, neglecting the effect of temperature variations, the value of Δf must be taken as one-half the value for full load, or $\Delta f = 0.281$ m.

C. The Arched Rib.

(Arch with Solid Web.)

§ 10. Internal Stresses in Curved Ribs.

We conceive an arched rib as generated by a plane figure whose center of gravity moves along a line of single curvature while the plane of the figure remains continually perpendicular to this directrix. The generating figure is called the cross-section of the arch and the directrix is called its axis.

It is presupposed that the lines of action of all the external forces acting on the arch (loads and end-reactions) lie in a single plane—the force-plane, which coincides with the plane of the arch-axis and contains a principal axis of each cross-section.

If the external forces are known, then the resultants of the internal stresses at any section are also determined, since these must be in equilibrium with all external forces acting on the rib between the given section and the end of the span. If the resultant of these forces at every section passes through the center of gravity of the cross-section, that is if the loads are so distributed that their equilibrium polygon coincides throughout with the arch-axis, then the stresses will be uniformly distributed over each cross-section and there will be none but normal stresses (pure tension or compression) throughout the structure. Under any other loading, for which the line of resultant pressures departs from the arch-axis, bending stresses will appear in the structure.

Upon applying any load, the axis of the rib assumes a new form called the elastic curve for the given loading. To be precise, the quantities employed in the following analysis ought to relate to the condition of the structure after deformation (cf. § 28). Nevertheless let us presuppose — what appears to be admissible in all practical cases—that the deformations are so slight that all relations based on the original coordinates of the axis will hold good, even after deformation.

We locate the origin of coordinates at the left end-point of the arch-axis and measure the positive abscissae toward the right and the positive ordinates upward. Let the inclination to the horizontal of the tangent to the arch-axis at the point (x, y), or the inclination to the vertical of the plane of the cross-section, be denoted by ϕ. Then, for a positive dx, if the curvature is convex upward, $d\phi$ must be taken as negative. Also let R denote the

resultant of the external forces acting on the portion of the arch to the left of the section (x,y). Let its components parallel to the coordinate axes be H and V, and the components parallel respectively to the tangent and normal to the arch-axis at the point (x, y) be P and S (Fig. 38). Let M denote the moment of the external forces about an axis perpendicular to the force-plane at the point (x, y), considered positive if directed clockwise. The external forces may then be replaced by a single force R, or its components (H,V) or (P, S) applied at the center of gravity of the given section together with a couple whose moment is M. With these external forces, the internal stresses at the section (x, y) must be in equilibrium.

Fig. 38.

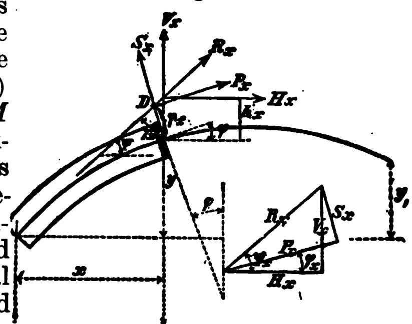
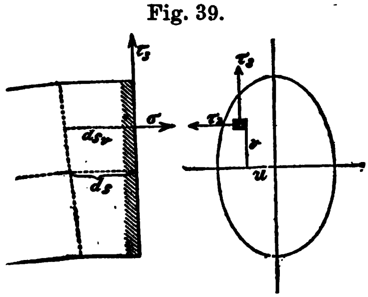


Fig. 39.



Consequently, if σ, τ_2 and τ_3 denote the normal and shearing stresses on any elementary area $dA = du.\,dv$ of the cross-section (Fig. 39), we may write the following equations of equilibrium:

$$\left.\begin{aligned} P+\int\sigma.dA &=0 \\ S+\int\tau_3.dA &=0 \\ \int\tau_2.dA &=0 \\ M+\int\sigma.v.dA &=0 \\ \int\sigma.u.dA &=0 \\ \int\tau_3.u.dA-\int\tau_2.v.dA &=0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots \quad (162.$$

1. Determination of the Normal Stress σ. It is assumed that the axis will remain a plane curve after deformation and that all bending takes place in the force-plane. The radius of curvature of the arch-axis at the point (x, y) will be denoted by r (*positive* if the curve is convex *upward*). If $d\,s_v$ denotes the length of a fiber between two cross-sections infinitely close together before deformation, and $\Delta\, ds_v$ is its change in length, then the length of the fiber after deformation will be $d\,s_v+\Delta\, ds_v$

We adopt the same assumption as in the common theory of flexure of straight beams, namely that the material cross-sections remain plane and perpendicular to the axis after deformation. By this assumption, if $-d\phi$ denotes the angle between the two consecutive cross-sections, and if this changes upon deformation into $-(d\phi+\Delta d\phi)$, ($d\phi$ and $\Delta d\phi$ being considered negative when the curve is convex upward and when the curvature is increased by the deformation), we have

$$ds_v = ds - v\,d\phi$$

and

$$ds_v + \Delta ds_v = ds + \Delta ds - v\,(d\phi + \Delta d\phi).$$

Subtracting the first of these equations from the second, we obtain:

$$\Delta ds_v = \Delta ds - v\cdot\Delta d\phi.$$

The proportional elongation or strain of the fiber is, therefore,

$$\frac{\Delta ds_v}{ds_v} = \frac{\Delta ds - v\Delta d\phi}{ds - v\,d\phi}$$

or, since $ds = -r\,d\phi$,

$$\frac{\Delta ds_v}{ds_v} = \left(\frac{\Delta ds}{ds} - v\,\frac{\Delta d\phi}{ds}\right)\frac{r}{r+v} \quad \ldots\ldots\ldots\ldots (163.$$

If, in addition to the normal stress, σ, there is also a temperature variation t contributing to the fiber-strain Δds_v, then, with a coefficient of expansion ω,

$$\frac{\Delta ds_v}{ds_v} = \frac{\sigma}{E} + \omega t.$$

Combining this with equ. (163) and solving for σ,

$$\sigma = E\left(\frac{\Delta ds}{ds} - v\,\frac{\Delta d\phi}{ds}\right)\frac{r}{r+v} - E\omega t \quad \ldots\ldots (164.$$

Substituting this expression in the first and fourth of the equations of equilibrium (162), we obtain

$$\left.\begin{aligned} P &= -E\frac{\Delta ds}{ds}\int\frac{r.dA}{r+v} + E\frac{\Delta d\phi}{ds}\int\frac{rv}{r+v}\,dA + E\omega t\,.\,A \\ M &= -E\cdot\frac{\Delta ds}{ds}\int\frac{rv}{r+v}\,dA + E\,\frac{\Delta d\phi}{ds}\int\frac{rv^2}{r+v}\cdot dA \\ &\quad + E\omega t\int v.dA \end{aligned}\right\} \ldots (165.$$

Observing that

$$\int dA = A, \quad \int v.dA = 0,$$

and hence that

$$\int \frac{r\,dA}{r+v} = \int dA - \int \frac{v\,dA}{r+v} = A - \frac{1}{r}\int v\,dA + \frac{1}{r}\int \frac{v^2 dA}{r+v}$$
$$= A + \frac{1}{r^2}\int \frac{r v^2}{r+v}\cdot dA$$

and

$$\int \frac{rv}{r+v}\,dA = \int v\,dA - \int \frac{v^2 dA}{r+v} = -\frac{1}{r}\int \frac{rv^2}{r+v}\,dA,$$

and introducing the abbreviation

$$\int \frac{r v^2}{r+v}dA = J, \quad \text{.......................... (166.}$$

then the equations (165) become

$$\left.\begin{array}{l} \dfrac{P}{E} = \left(-\dfrac{\Delta ds}{ds} + \omega t\right)A - \left(\dfrac{\Delta d\phi}{ds} + \dfrac{1}{r}\dfrac{\Delta ds}{ds}\right)\dfrac{J}{r} \\ \text{and} \\ \dfrac{M}{E} = \left(\dfrac{\Delta d\phi}{ds} + \dfrac{1}{r}\dfrac{\Delta ds}{ds}\right)J. \end{array}\right\}$$

Solving these equations, we find

$$\left.\begin{array}{l} -\dfrac{\Delta ds}{ds} = \dfrac{P}{EA} + \dfrac{M}{EAr} - \omega t \\ \dfrac{\Delta d\phi}{ds} = \dfrac{P}{EAr} + \dfrac{M}{EAr^2} - \dfrac{\omega t}{r} + \dfrac{M}{EJ} \end{array}\right\} \text{...... (167.}$$

Substituting these values in equ. (164), we obtain the normal fiber stress

$$-\sigma = \frac{P}{A} + \frac{M}{Ar} + \frac{Mrv}{J(r+v)} \quad \text{.............. (168}$$

This equation (168) shows that σ is not a linear function of v but, on the contrary, is represented by the ordinates of an equilateral hyperbola whose asymptote normal to the given section passes through the center of curvature.

The fiber stress σ will be zero at

$$v = -\frac{\dfrac{Pr}{A} + \dfrac{M}{A}}{\dfrac{P}{A} + \dfrac{M}{Ar} + \dfrac{Mr}{J}}$$

Hence, if $P=0$, the neutral axis does not pass through the center of gravity of the cross-section ($v=0$), unless $r=\infty$, i. e., unless the axis of the rib is originally straight.

The quantity J may be expanded into a series:

$$J = \int \frac{v^2 r}{r+v}dA = \int v^2 dA - \frac{1}{r}\int v^3\left(1 - \frac{v}{r} + \frac{v^2}{r^2} - \dots\right)dA.$$

But $\int v^2 \, dA$ represents the moment of inertia I of the cross-section about the U-axis. If, furthermore, the cross-section is symmetrical about this axis, all the terms in $v^3, v^5 \ldots$ will vanish, so that we obtain

$$J = I + \frac{1}{r^2} \int v^4 \left(1 + \frac{v^2}{r^2} + \frac{v^4}{r^4} + \ldots\right) dA.$$

If the radius of curvature r is very large in proportion to the depth of cross-section, all the other terms in the expression for J may be neglected in comparison with the term I, so that we may substitute the moment of inertia I for the quantity J in the formulae established above.

For example, in a rectangular cross-section of width b and depth h, we find:

for a ratio of $\frac{h}{r} =$	0.05	0.10	0.20
$\frac{J}{I} =$	1.0004	1.0015	1.0060

This approximation appears to be admissible in all practical applications of the formulae to arch-bridges. In most cases it even suffices to replace equs. (167) and (168) by the simpler equations applying to straight beams:

$$\left.\begin{aligned} -\frac{\Delta ds}{ds} &= \frac{P}{EA} - \omega t, \\ \frac{\Delta d\phi}{ds} &= \frac{M}{EI} \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (167^a.$$

$$-\sigma = \frac{P}{A} + \frac{Mv}{I} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (168^a$$

The fundamental relations employed in the preceding analysis impose one more test: to determine under what conditions, if any, the fifth of the equs. (162), also involving the quantity σ, will be satisfied. Substituting the expression of (164) and making a slight omission, we obtain

$$E\left(\frac{\Delta ds}{ds} - \omega t\right)\int u \, . \, dA - E\frac{\Delta d\phi}{ds}\int \frac{ruv}{r+v} \cdot dA = 0.$$

or, since $\int u \, . \, dA = 0$,

$$r \int \frac{uv \, . \, dA}{r+v} = 0.$$

This equation will be satisfied by the condition, among others, that the cross-section be symmetrically disposed about the force-plane; if r is very large compared with v, the equation reduces to

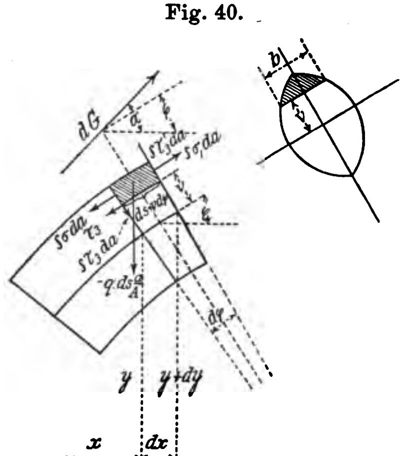
$$\int u\,v\,.\,dA = 0,$$

which shows that one of the principal axes of the section must lie in the force-plane.

2. Determination of the Radial Shearing Stress τ_3. As a rule, the shearing stresses in an arch may be neglected; nevertheless a knowledge of them is sometimes important, and the value of τ_3 should therefore be derived. We shall adopt the assumption, however, that *the radius of curvature is sufficiently large compared with the depth of cross-section* to permit the use of equ. (168ᵃ) in determining the normal stresses.

If, from the rib-segment included between the sections at (x, y) and $(x + dx, y + dy)$, we cut off an upper portion

Fig. 40.



by means of a section perpendicular to the force-plane and parallel to the arch-curve, there will be acting upon this elementary body the forces and stresses represented in Fig. 40.

Here $d\,G$ denotes the external load acting on the arch segment and $q\,d\,s$ the dead weight of the latter. It is assumed that the shearing stresses τ_3 are the same for any two mutually perpendicular planes at any point and that they are *uniform in intensity throughout the elementary tangential section.* Distinguishing the stresses at the neighboring section by affixing an index (′) and presupposing but a slight variation of cross-section, we have, by (168ᵃ),

$$\begin{cases} \int \sigma . dA = -\dfrac{P.a}{A} - \dfrac{M\Sigma}{I} \\ \int \sigma' . dA = -\dfrac{(P+dP)a}{A} - \dfrac{(M+dM)\Sigma}{I} \end{cases}$$

Here a denotes the area of cross-section included above the tangential cutting plane, and Σ denotes the static movement of this area about the U-axis. Now (by Fig. 38) we have

$$P = H \cos\phi + V \sin\phi, \qquad (169.$$

hence

$$\frac{dP}{ds} = (-H \sin\phi + V \cos\phi)\frac{d\phi}{ds} + \frac{dV}{ds}\sin\phi + \frac{dH}{ds}\cos\phi.$$

But, since (by Fig. 38),

$$S = -H \,.\, \sin\phi + V \cos\phi \qquad (170.$$

and

$$dV = dG \,.\, \sin\alpha - q\, ds, \quad dH = dG \cos\alpha, \quad \frac{d\phi}{ds} = -\frac{1}{r},$$

we obtain

$$\frac{dP}{ds} = -\frac{S}{r} + \frac{dG}{ds}\cos(\alpha - \phi) - q \sin\phi = -\frac{S}{r} + q_n, \qquad (171.$$

where q_n denotes the intensity (per unit length of axis) of the normal component of the total load on the elementary solid. In like manner we obtain

$$dM = -H \,.\, ds \,.\, \sin\phi + V \,.\, ds \,.\, \cos\phi + dG \,.\, m \,.\, \cos(\alpha - \phi);$$

hence,

$$\frac{dM}{ds} = S + m\frac{dG}{ds}\cos(\alpha - \phi), \qquad (172.$$

where m is the radial lever arm of the force dG about an axis through the point (x, y).

Finally, the condition that the algebraic sum of all the force-components parallel to the tangent must be zero, yields the following equation adapted for determining τ_s:

$$-\tau_s \,.\, b\,(ds - v\,d\phi) - d\phi\int \tau_s' \,.\, dA + \int \sigma' \,.\, dA - \int \sigma \,.\, dA$$
$$+ dG \cos(\alpha - \phi) - q \,.\, ds \,.\, \frac{a}{A} \cdot \sin\phi = 0.$$

Substituting $ds - v\,d\phi = -d\phi\,(r + v)$ or, by the above assumptions, $= -d\phi \,.\, r = ds$, putting $\int \tau'_s \,.\, dA = \int \tau_s \,.\, dA$, and substituting the value of $\int \sigma'\, dA - \int \sigma \,.\, dA$, also the expressions of (171) and (172), we obtain

$$\tau_s \,.\, b - \int \frac{\tau_s \,.\, dA}{r} = S\left(\frac{a}{A\,.\,r} - \frac{\Sigma}{I}\right) + \frac{dG}{ds}\cos(\alpha - \phi)\left(1 - \frac{a}{A} - \frac{m\,\Sigma}{I}\right).$$

If we put $-\frac{d\phi}{d\,s}=\frac{1}{r}=0$, the above reduces to

$$\tau_3 b = -\frac{S\Sigma}{I} + \frac{d\,G}{ds} \cos(\alpha-\phi)\left(1-\frac{a}{A}-\frac{m\,\Sigma}{I}\right) \ldots\ldots (173.$$

If, in addition, $\frac{d\,G}{ds}=0$ or $\cos(\alpha-\phi)=0$, the above becomes

$$\tau_3 = -\frac{S\,\Sigma}{b\,I} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (173^a.$$

The above results (equs. 168^a and 173^a) show that *the formulae for straight beams may be applied, with close approximation, to the arched rib, provided the radius of curvature is comparatively large*; in all further stress-analysis, determination of principal stresses, etc., the same principle will be applied and flat arches will be treated approximately as straight beams.

In bridge-arches for the usual descriptions of loading, the shear S will always be very small compared with the axial-force P, so that, as a rule, the shearing stresses in arched ribs may be omitted from consideration.

§ 11. Conditions for Stability. Line of Resistance. Core-Points. Graphic Determination of Normal Fiber Stresses.

Neglecting the shearing stresses, as just suggested, the principal stresses become identical with the normal fiber stresses. These, by equ. (168), attain their greatest values at the extreme fibers; if these fibers are distant v_1 and $-v_2$ from the gravity-axis, the respective stresses will be:

$$\left.\begin{aligned}\sigma_1 &= -\frac{P}{A} - \frac{M}{A\,r} - \frac{M\,r\,v_1}{J\,(r+v_1)}\\ \sigma_2 &= -\frac{P}{A} - \frac{M}{A\,r} + \frac{M\,r\,v_2}{J\,(r-v_2)}\end{aligned}\right\} \ldots\ldots\ldots\ldots (174.$$

Neither of these values should exceed the safe working intensity of stress.

If r is sufficiently large relative to v, the above expressions become

$$\left.\begin{aligned}\sigma_1 &= -\frac{P}{A} - \frac{M\,v_1}{I}\\ \sigma_2 &= -\frac{P}{A} + \frac{M\,v_2}{I}\end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (174^a.$$

which values may also be obtained directly from equ. (168^a).

If we substitute $M = P\,.\,p$, then it appears that the distribution of fiber stresses in the cross-section is especially dependent upon the value of p, i. e., upon the distance of the piercing point D (Fig. 38) of the external resultant R from the gravity-axis. It is therefore desirable to know the position of this point at every cross-section; and, for any given loading, the curve constituting the locus of these piercing points is named the *Line of Resistance.*

The curve enveloping the successive resultants of the external forces (generally differing but little from the line of resistance), i. e., the equilibrium curve of the external loading, is called the *Line of Pressures* or, more briefly, the *Pressure Line.* A line drawn from any point of the line of resistance tangent to the pressure line will constitute the line of action of the corresponding resultant pressure R. The name, line of resistance, is adopted because that line determines at each section the point at which the internal resisting stresses must balance the external forces to produce stable equilibrium. Another important property of the line of resistance is that its dis-

tance from the gravity axis controls the magnitude and distribution of the stresses in the cross-section.

If i denotes the radius of gyration of the cross-section in the force-plane, then $I = A \cdot i^2$, and by equ. (168^a),

$$\sigma = -\frac{P}{a} - \frac{Mv}{I} = -\frac{P}{a}\left(1 + \frac{pv}{i^2}\right) \ldots\ldots\ldots \quad (175.$$

Equ. (175) shows that the normal stresses vary along the section as the ordinates of a straight line; specifically,

$$\left.\begin{array}{ll} \text{at } v = v_1, & \sigma_1 = -\frac{P}{A}\left(1 + \frac{p\,v_1}{i^2}\right) \\ \text{at } v = 0, & \sigma_0 = -\frac{P}{A} \\ \text{at } v = -v_2, & \sigma_2 = -\frac{P}{A}\left(1 - \frac{p\,v_2}{i^2}\right) \end{array}\right\} \ldots\ldots \quad (176.$$

Equations (176) show that the extreme stresses σ_1 and σ_2, and hence all the stresses in the section, will have the same sign provided

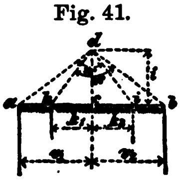
$$p > -\frac{i^2}{v_1} \text{ and, at the same time, } < \frac{i^2}{v_2} \ldots\ldots \quad (177.$$

Hence, in each cross-section, if we fix two points at distances k_1 and $-k_2$ from the gravity axis, determined by

$$k_1 = \frac{i^2}{v_2}, \qquad k_2 = \frac{i^2}{v_1} \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (178.$$

or by the corresponding graphic construction of Fig. (41), they will so divide the section that all the normal stresses will be of the same sign so long as the line of resistance, representing the resultant pressure R, passes between the two points. These points, whose position depends only upon the form of the cross-section, are called the *Core-Points,* their continuous loci are the two *Core-Lines,* and the portion of the force-plane intercepted between these lines is called the *Core.*

Fig. 41.



All the normal fiber stresses at any section will therefore have the same sign or direction, so long as the line of resistance cuts the section within the core.

Introducing the quantities k_1 and k_2 into equations (176), we obtain

$$\left.\begin{array}{l} \sigma_1 = -\frac{P}{A}\left(\frac{k_2 + p}{k_2}\right) = -\frac{P\,(k_2 + p)}{I} \cdot v_1 = -\frac{M_{k_2} v_1}{I} \\ \sigma_2 = -\frac{P}{A}\left(\frac{k_1 - p}{k_1}\right) = -\frac{P\,(k_1 - p)}{I} \cdot v_2 = +\frac{M_{k_1} v_2}{I} \end{array}\right\} \ldots \quad (179.$$

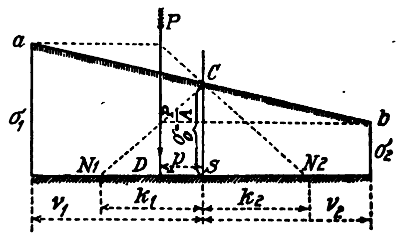
where M_{k_1} and M_{k_2} represent the moments of the external forces about the two core-points of the cross-section.

These equations, or the equivalent forms

$$\left.\begin{aligned}\sigma_1 &= -\frac{P}{A}\left(1+\frac{p}{k_2}\right)\\ \sigma_2 &= -\frac{P}{A}\left(1-\frac{p}{k_1}\right)\end{aligned}\right\}, \qquad (179^a.$$

form the basis for the graphic method of Fig. 42 for determining the normal stresses. N_1, N_2, are the core-points lying in the force-plane; and the ordinate $S\,C$, at the center of gravity, is made equal to $\frac{P}{A}$. The lines drawn from the core-points N_1, N_2, to the point C will intercept on the resultant normal pressure P, two lengths representing extreme fiber stresses σ_1 and σ_2 respectively.

Fig. 42.



From this construction it is evident that the stress at one of the extreme fibers will be zero when the line of resistance passes through the corresponding core-point; that the stresses σ_1 and σ_2 will have like signs when the resultant force passes inside the core, and unlike signs when it passes outside the core; and the neutral axis will come the nearer to the center of gravity the farther the line of resistance is removed therefrom. With the aid of analytical geometry it may also be shown that the neutral axis is the anti-polar of the central ellipse with respect to the point of application of the resultant pressure as pole.

If the cross-section consists of two *flanges* of area A_1 and A_2, and if h is the effective depth or distance between the centers of gravity of the flanges, then approximately, neglecting the effect of the web, we may write $A_1 v_1 = A_2 v_2$, $A_1 v_1^2 + A_2 v_2^2 = (A_1+A_2)\,i^2$, and $k_1 v_2 = i^2$, $k_2 v_1 = i^2$; hence we find: $k_1 = v_1$, $k_2 = v_2$, i. e., the core-points coincide with the flanges of the girder. Consequently, by equ. (179),

$$\left.\begin{aligned}\sigma_1 &= -\frac{P\,(p+k_2)}{A_1\,(k_1+k_2)} = -\frac{M_l}{A_1 h}\\ \sigma_2 &= -\frac{P\,(k_1-p)}{A_2(k_1+k_2)} = +\frac{M_u}{A_2 h}\end{aligned}\right\} \qquad (180.$$

Here M_u and M_l represent the moments of the external forces taken about the centers of gravity of the upper and lower chords respectively.

§ 12. Determination of the Deformations.

1. Elongation of the Axis. If (x, y) and (x_0, y_0) are the co-ordinates of two points on the arch axis and if s and s_0 are their respective distances measured along the axis from any assumed initial point, the change in the axial distance between the two points is obtained by integrating the first of equs. (167); there results:

$$\Delta s - \Delta s_0 = \omega t (s - s_0) - \int_{s_0}^{s} \left(\frac{P}{EA} + \frac{M}{EAr}\right) ds \ldots (181.$$

and approximately, if r is very large,

$$\Delta s - \Delta s_0 = \omega t (s - s_0) - \int_{s_0}^{s} \left(\frac{P}{EA}\right) ds \ldots\ldots (181^a.$$

2. Variation of the Angle ϕ. The variation of the angle between the normals or tangents at two points whose initial coordinates are (x, y) and (x_0, y_0) is given by the integration of the second of equs. (167):

$$\left.\begin{aligned}\Delta\phi - \Delta\phi_0 &= \int_{s_0}^{s} \left(\frac{P}{EAr} + \frac{M}{EAr^2} + \frac{M}{EJ}\right) ds - \omega t \int_{s_0}^{s} \frac{ds}{r} \\ &= \int_{s_0}^{s} \frac{M}{EJ}\, ds - \frac{\Delta s - \Delta s_0}{r}\end{aligned}\right\} \ldots\ldots (182.$$

If r is very large compared with the dimensions of the cross-section, then approximately,

$$\left.\begin{aligned}\Delta\phi - \Delta\phi_0 &= \int_{s_0}^{s} \frac{Mds}{EI} - \omega t \int_{s_0}^{s} \frac{ds}{r} \\ &= \int_{s_0}^{s} \frac{Mds}{EI} + \omega t (\phi - \phi_0)\end{aligned}\right\} \ldots\ldots (182^a.$$

3. Variation of the Radius of Curvature. If r_1 is the radius of curvature at any point after deformation, then

$$\frac{1}{r_1} = -\frac{d\phi + \Delta d\phi}{ds + \Delta ds},$$

so that the variation of curvature is

$$\frac{1}{r_1}-\frac{1}{r}=-\frac{d\phi+\Delta d\phi}{ds+\Delta ds}+\frac{d\phi}{ds}=-\frac{\frac{\Delta d\phi}{ds}+\frac{1}{r}\frac{\Delta ds}{ds}}{1+\frac{\Delta ds}{ds}}.$$

On substituting the values from equs. (167), the above becomes

$$\frac{1}{r_1}-\frac{1}{r}=-\frac{M}{EJ}\cdot\frac{1}{\left(1-\frac{P}{EA}-\frac{M}{EAr}+\omega t\right)} \quad \ldots\ldots (183.$$

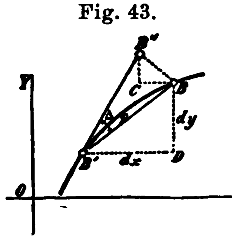
As an approximation, since $\frac{\Delta ds}{ds}$ is always very small compared with unity and as r is assumed very large relative to the sectional dimensions, we may write

$$\frac{1}{r_1}-\frac{1}{r}=-\frac{M}{EI} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (183^a.$$

If the arch is made to conform to an equilibrium curve, then, as the line of resistance coincides with the axis, $M=0$, and, if the ends are free to undergo the necessary displacements, by equ. (183), $\frac{1}{r_1}=\frac{1}{r}$, or the radius of curvature remains constant at every point. The deformation then consists merely of a shortening of the axis by the distance Δds. Consequently there must be, on the whole, a shortening of the span. As this, however, is impossible with an arch whose ends are immovable, it is seen that a perfect coincidence of arch-axis with line of resistance cannot be attained, or may be assumed approximately only when the deformations are negligible or the material is considered incompressible.

4. Variation of the Coordinates. Let Δx, Δy denote the displacements of the point B (x, y) in the directions of the respective coordinate axes (Fig. 43); and let Δdx, Δdy, Δds represent the variations in the distances dx, dy, ds between the two consecutive points, B, B'. Let the section at B' undergo a rotation $\Delta\phi$ which we assume equal to that at B. The point B of the arch-axis is thus displaced to B''; and the projections of this displacement BB'' upon the coordinate axes may be determined from the triangle $BB''C$ which is similar to the triangle $BB'D$. We thus have

Fig. 43.



$$\begin{cases} BC=-\Delta\phi\,.\,ds\,.\,\frac{dy}{ds}=-\Delta\phi\,.\,dy, \text{ and} \\ B''C=\Delta\phi\,.\,ds\,.\,\frac{dx}{ds}=\Delta\phi\,.\,dx\,. \end{cases}$$

If the arch-axis suffers, in addition, a change in length, in particular if the arch element $BB'=ds$ is lengthened by Δds,

then the point B'' receives a displacement whose rectangular components are $\Delta\, ds \frac{dx}{ds}$ and $\Delta\, ds \frac{dy}{ds}$, so that the total displacement of B relative to B' is given by the expressions,

$$\Delta dx = -\Delta\phi \,.\, dy + \Delta ds \,.\, \frac{dx}{ds}$$

$$\Delta dy = \Delta\phi \,.\, dx + \Delta ds \,.\, \frac{dy}{ds}$$

Considering such displacements in all the arch elements beginning at some initial point A, and summing these up to get the total displacement of B, we obtain,

$$\left.\begin{aligned} \Delta x &= -\int \Delta\phi \,.\, dy + \int \frac{\Delta\, ds}{ds} \cdot dx \\ \Delta y &= \int \Delta\phi \,.\, dx + \int \frac{\Delta\, ds}{ds} \cdot dy \end{aligned}\right\}$$

If we indicate the values of the coordinate and angular variations pertaining to the initial point A with the subscript 0, and the values pertaining to the point B with the subscript 1, then, by partial integration of the above equations and substituting the proper limits, we obtain

$$\left.\begin{aligned} \Delta x_1 - \Delta x_0 &= -\Delta\phi_1 y_1 + \Delta\phi_0 y_0 + \int_{s_0}^{s_1} \frac{\Delta d\phi}{ds} \,.\, y ds + \int_{x_0}^{x_1} \frac{\Delta ds}{ds} dx \\ \Delta y_1 - \Delta y_0 &= \Delta\phi_1 x_1 - \Delta\phi_0 x_0 - \int_{s_0}^{s_1} \frac{\Delta d\phi}{ds} \,.\, x ds + \int_{x_0}^{x_1} \frac{\Delta ds}{ds} dy \end{aligned}\right\} \quad (184.$$

Substituting for $\frac{\Delta d\phi}{ds}$ and $\frac{\Delta ds}{ds}$ the expressions (167), but introducing the abbreviation $P + \frac{M}{r} = P'$, and writing $\Delta\phi_1 = \Delta\phi_0 + \int_{s_0}^{s_1} \frac{\Delta d\phi}{ds} \,.\, ds$, we obtain,

$$\left.\begin{aligned} &\Delta x_1 - \Delta x_0 = -\Delta\phi_0 (y_1 - y_0) - \int_{s_0}^{s_1} \frac{M}{EJ} (y_1 - y)\, ds \\ &- \int_{s_0}^{s_1} \frac{P'}{EA} \left(\frac{y_1 - y}{r} ds + dx \right) + \omega t \int_{s_0}^{s_1} \left(\frac{y_1 - y}{r} \,.\, ds + dx \right) \end{aligned}\right\} \ldots (185$$

$$\left.\begin{aligned}\Delta y_1-\Delta y_0=\Delta\phi_0(x_1-x_0)+\int_{s_0}^{s_1}\frac{M}{EJ}(x_1-x)\,ds\\+\int_{s_0}^{s_1}\frac{P'}{EA}\Big(\frac{x_1-x}{r}ds-dy\Big)-\omega t\int_{s_0}^{s_1}\Big(\frac{x_1-x}{r}ds-dy\Big)\end{aligned}\right\}\ldots\ldots(186.$$

The above equations may be simplified if the radius of curvature is very large relative to the sectional dimensions. We may then put $P'=P$, $J=I$, and, in the temperature terms, omit the parts containing $\frac{1}{r}$. There results:

$$\left.\begin{aligned}\Delta x_1-\Delta x_0=-\Delta\phi_0(y_1-y_0)-\int_{s_0}^{s_1}\frac{M}{EI}(y_1-y)ds\\-\int_{s_0}^{s_1}\frac{P}{EA}\Big(\frac{y_1-y}{r}\cdot ds+dx\Big)+\omega t(x_1-x_0)\;.\end{aligned}\right\}\ldots(185^a.$$

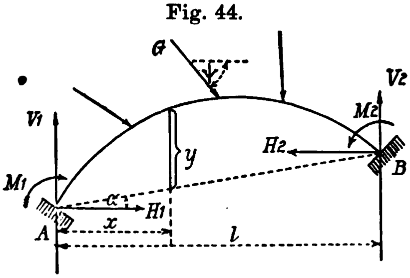
$$\left.\begin{aligned}\Delta y_1-\Delta y_0=\Delta\phi_0(x_1-x_0)+\int_{s_0}^{s_1}\frac{M}{EI}(x_1-x)ds\\+\int_{s_0}^{s_1}\frac{P}{EA}\Big(\frac{x_1-x}{r}ds-dy\Big)+\omega t(y_1-y_0)\;.\end{aligned}\right\}\ldots(186^a.$$

§ 13. External Forces of the Arched Rib.

We here consider plate arches or curved ribs whose ends are connected to rigid supports. The connections may be either such as to keep the ends of the rib completely or practically immovable, although permitting free rotation of the axis—*Arches with Hinged Ends*—or such as to prevent rotation also, so that the ends of the axis remain fixed in direction—*Arches with Fixed Ends (or Hingeless Arches).* Both of these types of arches, as explained in the Introduction (§ 1), belong to the statically indeterminate class of structures, i. e., the conditions of static equilibrium do not suffice to determine the external reactions; it thus becomes necessary to establish additional equations of condition on the basis of the elastic deformations. For this purpose we may employ the deformation equations established in the preceding section (§ 12), or, as will later be shown, we may apply the Theorem of Least Work. Static determinateness, however, may be attained in the arch with hinged ends by introducing a third hinge, thus obtaining the *Three-Hinged Arch.*

In the general case of the fixed-ended, hingeless arch, there exist the following relations between the external forces: Let A and B (Fig. 44) be the two end points, assumed to be at different elevations. Let G represent one of the forces acting in any direction upon the arch. Let us, for the moment, imagine the end A free to slide horizontally while the end B, at the same time, has a hinged connection; and, in the freely supported girder thus obtained, let us determine the bending moment **M** at any point (x, y), the ordinate y being measured from the closing chord $A\,B$.

Fig. 44.



If g and γ are the lever arms of the external forces about the end B and the point (x, y), respectively, then

$$\mathbf{M} = \frac{x}{l}\sum_0^l G\,.\,g - \sum_0^x G\,.\,\gamma \ldots\ldots\ldots\ldots\ldots\ldots \quad (187.$$

The fixedness of the ends of the rib, preventing either rota-

tion or horizontal displacement, may be replaced by the effect of a force with horizontal component H_1, applied at A and acting in the direction of the chord $A\,B$, and a moment M_1 applied at A together with another M_2 applied at B; these reactions must be added to those of a freely supported girder. The bending moment at the point (x, y) then becomes:

$$M = \mathrm{M} + \frac{M_1(l-x)+M_2.x}{l} - H_1 \cdot y \quad \ldots\ldots\ (188.$$

Using the notation indicated in Fig. 44, we may also write the following general equations:

$$\left.\begin{aligned} V_1 &= \frac{1}{l}\sum_0^l G.g + \frac{M_2 - M_1}{l} + H_1 \tan a \\ -V_2 &= V_1 - \sum_0^l G \sin\psi \\ H_2 &= H_1 + \sum_0^l G \cos\psi \end{aligned}\right\} \quad \ldots.\ (189$$

At any section x, the horizontal and vertical components will be

$$\left.\begin{aligned} H &= H_1 + \sum_0^x G.\cos\psi \\ V &= V_1 - \sum_0^x G.\sin\psi \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ (190.$$

If ϕ is the slope of the tangent to the arch-axis at the point (x, y), the axial and shearing components are given by

$$\begin{cases} P = V \sin\phi + H \cos\phi \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (191. \\ S = V \cos\phi - H \sin\phi \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (192. \end{cases}$$

With M and P known, we may determine the internal stresses in the arch by equ. (168) or (168^a), and the maximum fiber stresses by equ. (174) or (174^a), so that the entire problem is solved. But in order to find M and P, in the general case, it is necessary to first determine three statically indeterminate quantities: H_1, M_1 and M_2. In the two-hinged arch the end moments M_1 and M_2 vanish; hence the number of indeterminate quantities reduces to one, viz., the horizontal thrust H_1. Finally, if a third hinge is put into the arch, it provides another static condition, namely that the moment of the external forces about the center of the hinge must vanish; this condition determines the value of H_1.

If the two ends are at the same level, and if all the applied loads are vertical in direction, the above general equations reduce to

$$H_2 = H = H_1\ , \quad V_1 = \frac{1}{l}\sum_0^l G.g + \frac{M_2 - M_1}{l} \quad \ldots\ldots\ (193.$$

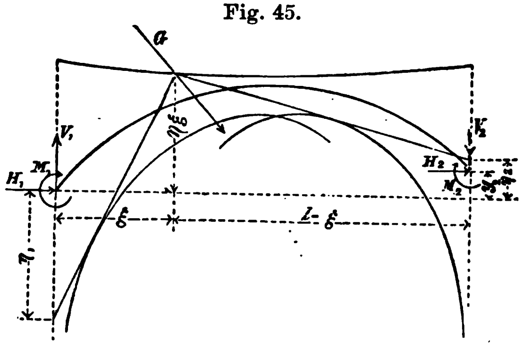
If the loading is *vertical*, **M** may be represented by the ordinates of a funicular polygon. If this polygon is constructed with the pole distance $= H_1$, and placed in such a position relative to the arch as to intersect the end-verticals at distances $e_1 = \frac{M_1}{H_1}$ and $e_2 = \frac{M_2}{H_1}$ above the points of support, then the ordinates of the polygon measured above the arch-chord and multiplied by H_1 will represent the moments $\mathbf{M} + \frac{M_1(l-x) + M_2 \cdot x}{l}$; consequently, by equ. (188), *the vertical intercepts between the equilibrium polygon of the loading and the axis of the arch, multiplied by H_1, will give the bending moments at the corresponding points of the axis.* This principle was first established by Winkler.

§ 14. Reaction Locus and Tangent Curves. Critical Loading.

1. Reaction Locus and Tangent Curves. When the end reactions, i. e., the resultants of (H_1, V_1) and (H_2, V_2), and the end moments, M_1 and M_2, are known for any given loading, we may construct the force polygon and equilibrium polygon and the line of resistance for that loading. The two end reactions intersect each other in a point on the resultant of the applied loads, which changes as the positions of the loads change; when this load variation follows some definite rule, the curve described by the intersection of the reactions is named the *Reaction Intersection Locus* or, more simply, the *Reaction Locus.*

With the shifting of the load resultant, the end reactions alter their position; one way in which these reactions may be determined is to have for each point of the reaction locus the

Fig. 45.



two points in which the reactions intersect the vertical lines passing through the ends of the arch-axis.

Another way of fixing the positions of the end reactions is by means of the two curves which are enveloped by them when the load resultant passes through the successive points of the reaction locus. The reactions are thus determined in position and direction, for any load, by the two tangents drawn from the corresponding point of the reaction locus to the two nappes of the enveloping curve. The magnitudes of the reactions are then found from the force triangle composed of the load-resultant and lines parallel to the two tangent directions. The two curves enveloping the reactions are called the *Reaction Envelope Curves* or, more simply, the *Tangent Curves.*

In Fig. 45, let (x, η_x) be the coordinates of any point of one

of the reaction-lines, (η_1, η_2) the distances intercepted by the reaction-lines on the left and right end-verticals respectively, and (ξ, η_ξ) the coordinates of the point of intersection of the two reaction-lines; then, in the general case,

$$\eta_1 = \frac{M_1}{H_1} \quad \ldots\ldots (194.$$

and

$$\eta_2 - y_2 = \frac{M_2}{H_2}, \text{ or } \eta_2 = y_2 + \frac{M_2}{H_2} \quad \ldots\ldots (195.$$

Also, for the left reaction,

$$\eta_{x<\xi} = \eta_1 + \frac{V_1}{H_1} \cdot x = \frac{M_1 + V_1 x}{H_1}, \quad \ldots\ldots (196.$$

and for the right reaction,

$$\eta_{x>\xi} = \eta_2 + \frac{V_2}{H_2}(l - x) = \frac{M_2 + H_2 y_2 + V_2 (l - x)}{H_2} \quad \ldots (197.$$

Putting $x = \xi$ in equ. (196) or (197), we obtain the ordinate of the reaction intersection point, or, with variable ξ, the *equation of the reaction locus:*

$$\eta_\xi = \frac{M_1 + V_1 \xi}{H_1} = \frac{M_2 + H_2 y_2 + V_2 (l - \xi)}{H_2} \quad \ldots\ldots (198.$$

To obtain the equation of the tangent curve, the value of ξ must be considered as the variable parameter in equations (196) and (197). The equation of the *tangent curve for the left reactions* is derived by eliminating ξ from equ. (196) $\left(\eta_{x<\xi} = \frac{M_1 + V_1 x}{H_1} \right)$ and the equation obtained by differentiating this with respect to ξ.

Similarly, we derive the equation of the *tangent curve of the right reactions* by eliminating ξ from equ. (197) $\left(\eta_{x>\xi} = \frac{M_2 + H_2 y_2 + V_2 (l - x)}{H_2} \right)$ and the equation obtained by differentiating this with respect to ξ.

If *end hinges* are considered, $M_1 = M_2 = 0$, and hence $\eta_1 = 0$, $\eta_2 = y_2$, or the reactions always pass through the centers of gravity of the end-sections; consequently the tangent curves in this case reduce to the two end-points. The reaction locus in this case will be

$$\eta_\xi = \frac{V_1}{H_1}\xi = \frac{H_2 y_2 + V_2 (l - \xi)}{H_2} \quad \ldots\ldots (198^a.$$

In the case of the *three-hinged arch,* the above formulae

apply in a general way. We may, however, make use of the relation that the line of resistance must, in this case, always pass through the three hinges, so that it is readily determined for any given loading. For a moving concentration, the reaction locus will then consist of two straight lines, meeting at the middle hinge, which are the prolongations of the lines joining the end hinges to the middle hinge.

On the basis of the above principles it is readily seen how the reactions may be found graphically as soon as the reaction locus and tangent curves are known. It should be observed that with the aid of the curves drawn for a single concentration we may construct not only the reactions for different positions of this load, but also those for a train of concentrations resting on the structure. These procedures become particularly simple when the ends are hinged since, in that case, all the reactions for the individual loads pass through the same point so that the position of the resultant reaction, determined in magnitude and direction by the corresponding force polygon, is at once known; whereas in hingeless arches this line of action must be determined by a special funicular polygon.

The reaction corresponding to a train of concentrations may be determined either as the resultant of the sums of the horizontal and the vertical components of the reactions for the individual loads, or as the closing side of a force polygon, whose sides represent the reactions for the individual loads. Next, combining the reaction determined by either of the above methods with the total loads acting on the portion of the rib separated from the rest by a given section, or by constructing the corresponding equilibrium polygon, we obtain the magnitude and position of the resultant pressure R at the section considered and with these data, as previously indicated, we may calculate the values of the extreme fiber stresses.

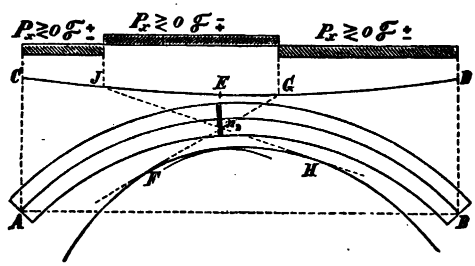
With the aid of the reaction locus and tangent curves drawn for a single concentration, the critical loading for any stress, i. e., the manner of loading for making the stress a maximum, may be determined.

2. Laws of Loading for Normal Stresses. As previously shown (§ 11), the normal fiber stresses attain their maximum values in the fibers farthest from the axis. Hence it will generally suffice to determine the critical loads just for these extreme fiber stresses; and it should here be recalled that the stress σ_1 in the *uppermost* fiber will have a sign the same as or opposite to that of the normal pressure P according as the resultant pressure R intersects the section above or below the *lower* core-point. On the other hand, the stress σ_2 in the *lowest* fiber will have its sign the same as or opposite to that of P, according as R cuts the section below or above the *upper* core-point. Furthermore, for a concentration G, the resultant R coincides with the first reaction R_1 if the load is located in the segment $(l - x)$, but with the equilibrant of the second reaction R_2 if the load is located in the segment (x). The sign of the normal pressure P for different positions of the load is thus dependent upon the signs of the individual values of R_1, R_2, and upon their position relative to the cross-section. For the

cases under present consideration, it may be stated that P will have the same sign for all sections and all load-positions; in fact, for vertical loading, P will be positive if the axis is concave downward (arch), and negative if the latter is convex downward (suspension cable).

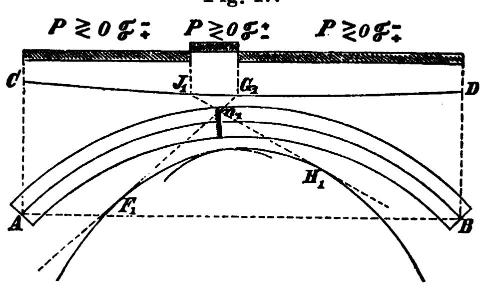
To investigate the effect of the loading on the *normal stresses in the uppermost fibers* of a cross-section (Fig. 46), we draw

Fig. 46.



two lines from the lower core-point n_2 of the section tangent to the tangent curves. The points J and G, in which these tangent rays cut the reaction locus, determine the *limits of loading* or "*critical points.*" Any load applied between C and J yields a resultant pressure cutting the section below n_2, hence producing tension (if P is positive); any load between J and E (the intersection of a vertical through the uppermost fiber with the reaction locus) yields a resultant pressure acting above $J\,H$, and every load between E and G yields a resultant above $F\,G$,

Fig. 47.



so that, in either case, the resultant will pass above n_2, producing compression in the upper fibers; every load between G and D yields a resultant pressure acting below $F\,G$ or n_2, producing tension in the given fibers.

For the *normal stresses in the lowest fibers* of a cross-section (Fig. 47), the tangents $J_1\,H_1$ and $F_1\,G_1$ must be drawn through the upper core-point n_1; the points $J_1\,G_1$ are then the critical points, so that all loads lying between C and J_1, or G_1 and D w

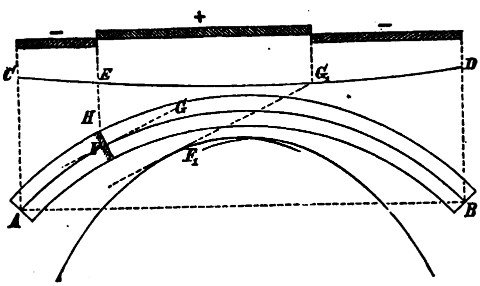
produce compression if $P > 0$, and all loads between J_1 and G_1 will produce the reverse stresses.

If, from the points C and D, lines are drawn tangent to the tangent curves, their intersections with the core-lines will divide the rib into portions having alternately one or two critical points, according as the intersections of the tangents from the core-points with the reaction locus fall within or without the limits of the span.

In the two-hinged arch, the end points take the place of the tangent curves; and, in the three-hinged arch, the reaction locus consists of two straight lines meeting at the middle hinge.

3. Laws of Loading for Shears S. The shears S determine the shearing stresses in the plate-arch or the stresses in the web members in the framed arch with parallel chords It is therefore important to determine the critical values of the shears.

Fig. 48.



Drawing the vertical line HE (Fig. 48) through the upper fiber H of the section and, parallel to the arch-tangent FG another line $F_1 G_1$ tangent to the tangent curve, the intersections E and G_1 with the reaction locus give the division points of the loading or "critical points" for shearing stress. For a section to the left of the crown, all loads between C and E, or G_1 and D, produce a negative shear, and all loads between E and G_1 produce a positive shear. For sections to the right of the crown, the reverse rule obtains; and, in general, the loading beginning at any section will have a positive or negative effect according as it extends toward the right or the left, and the limits of the loading are given either by the end of the rib or by the intersection of the reaction locus with the parallel tangent.

Drawing tangents from the points C and D to the tangent curves, the points at which the arch axis is parallel to these tangents divide the rib into portions having alternately one or two critical points. Naming the points of contact of the tangents under consideration M and N, then for all sections between M and N the vertical through the uppermost fiber gives the sole critical point, since the intersection G_1, used above, here falls outside the span. The extreme values of the shear S will then be produced by loads extending from the given section to the left or right ends of the span, exactly as in the case of maximum shears in a simple truss.

1. THE THREE-HINGED ARCH.

§ 15. External Forces.

By equ. (188), if the ends of an arched rib are hinged, so that $M_1 = M_2 = 0$, we have in general

$$M = \mathbf{M} - H_1 y \quad \ldots\ldots\ldots\ldots\ldots\ldots (199.$$

The insertion of a third hinge at the point ($x = g$, $y = f$) of the arch-axis supplies an additional equation of condition enabling H_1 to be determined directly. Namely, at the hinge point the moment of the external forces must vanish, and hence

$$\mathbf{M}_0 - H_1 f = 0$$

where $\mathbf{M}_0$ is the simple beam bending moment at the center of the hinge. Consequently

$$H_1 = \frac{\mathbf{M}_0}{f} \quad \ldots\ldots\ldots\ldots\ldots\ldots (200.$$

If a vertical load G is applied at a distance ξ ($< g$) from the end of the arch, we have $\mathbf{M}_0 = G \cdot \frac{\xi}{l}\ (l - g)$, and therefore

$$H = G \cdot \frac{\xi (l - g)}{l . f} \quad \ldots\ldots\ldots\ldots\ldots\ldots (201$$

Similarly we find, for $\xi > g$,

$$H = G \frac{(l - \xi) . g}{l . f} \quad \ldots\ldots\ldots\ldots\ldots\ldots (201^a.$$

The influence line for horizontal thrust (H) in a three-hinged arch thus proves to be a triangle with base l, whose apex lies on the vertical line through the middle hinge and whose altitude is equal to $G \cdot \frac{g (l - g)}{l . f}$.

If the middle hinge lies at the crown, then $g = \frac{l}{2}$, so that the horizontal thrust for a unit load is given by equs. (201) and (201^a), as

$$H = G \frac{\xi}{2 f}, \text{ or } H = G \frac{(l - \xi)}{2 f}.$$

With the value of H known, we may determine the moment M for any load applied at $x = \xi$, by equ. (199):

$$\left.\begin{aligned} M_{x<\xi} &= G\frac{l-\xi}{l}\cdot x - H\cdot y \\ M_{x>\xi} &= G\frac{\xi}{l}(l-x) - H\cdot y \end{aligned}\right\} \dots\dots\dots\dots \text{(202.}$$

We also obtain the following expressions for the axial and transverse forces from equs. (191) and (192):

$$\left.\begin{aligned} P_{x<\xi} &= G\frac{l-\xi}{l}\sin\phi + H\cos\phi \\ P_{x>\xi} &= -G\frac{\xi}{l}\sin\phi + H\cos\phi \end{aligned}\right\} \dots\dots\dots\dots \text{(203.}$$

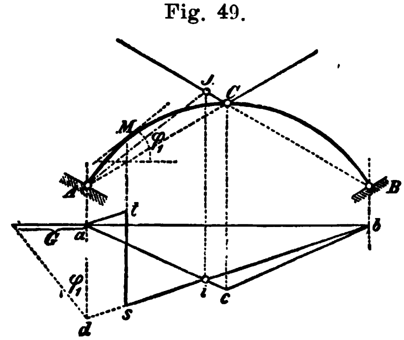
$$\left.\begin{aligned} S_{x>\xi} &= G\frac{l-\xi}{l}\cos\phi - H\sin\phi \\ S_{x<\xi} &= -G\frac{\xi}{l}\cos\phi - H\sin\phi \end{aligned}\right\} \dots\dots\dots\dots \text{(204.}$$

It is advantageous to take the moments M about the core-points of the cross-section, substituting for y the ordinates of these points, since, by equ. (179), the extreme fiber stresses are proportional to these moments; it then becomes unnecessary to proceed with the determination of the axial forces.

The moment and shear influence lines are formed of broken straight lines whose vertices lie in the verticals of the crown and of the given section, and whose equations are given by (202) and (204) if ξ is considered the variable. These influence lines may also be obtained by a simple construction which, for the moments, coincides with that of Fig. 32 if M_u and M_1 occurring there, are understood to represent the core-points of the given section.

The shear influence line is drawn in Fig. 49. The two parallel directions $a\,t$ and $b\,s$ are determined by the intercept $a\,d = G\,.\cot\phi_1$ on the end-verticals; furthermore the point i must lie vertically below J, the point at which the reaction locus is intersected by a line drawn from the end hinge parallel to the section tangent. The intercepts of the figure included between $a\,t\,s\,b$ and $a\,c\,b$ give the values of the shear for a unit load $= \dfrac{G}{\sin\phi_1}$.

Fig. 49.



The horizontal thrust for a uniform load covering the entire span, with an intensity of p per unit length, is

$$H_{tot}=\frac{pl^2}{8f} \quad \text{.......................... (205.}$$

If the load extends only for a distance λ_1 from one end, then, by equ. (200),

$$\left.\begin{array}{ll} \text{for } \lambda_1<\frac{l}{2}, & H=\frac{p\lambda_1^2}{4f} \\ \lambda_1>\frac{l}{2}, & H=\frac{p}{8f}\,(4\lambda_1 l-2\lambda_1^2-l^2) \end{array}\right\} \quad \text{......... (206.}$$

By the general rules given in § 14 it is a simple matter to determine, for any given form of arch, the greatest moments and shears producible by a moving load. The influence lines may be used for this purpose, this method being advantageous whenever the load consists of a train of concentrations.

For a continuous uniform load, the critical loading for moments is determined by:

$$\frac{\lambda_1}{l}=\frac{2fx_k}{2fx_k+ly_k} \quad \text{......................... (207.}$$

where x_k and y_k are the coordinates of the appropriate core-point in the given section (cf. equ. 98). If the distance $\lambda_1<\frac{l}{2}$ from the left end is covered with the uniform load p, we have

$$\left.\begin{array}{rl} M_{max}= & \frac{p\lambda_1 \cdot x_k(2l-\lambda_1)}{2l}-p\frac{x_k^2}{2}-p\frac{\lambda_1^2}{4f}\cdot y_k \\ = & \frac{1}{2}\,p\,x_k\,(\lambda_1-x_k) \end{array}\right\} \quad \text{..... (208.}$$

which is identical with the moment in a simple beam of span λ_1.

If the distance $(l-\lambda_1)$ is covered with the load p, we obtain

$$\left.\begin{array}{rl} M_{min}= & \frac{1}{2}\,p\,x_k\,(l-x_k)-\frac{pl^2}{8f}\cdot y_k-M_{max} \\ = & \frac{1}{2}\,p\,x_k\,(l-\lambda_1)-\frac{pl^2}{8f}\cdot y_k \end{array}\right\} \quad \text{... (209.}$$

The critical point for maximum shears is given by

$$\frac{\lambda_2}{l}=\frac{2f}{2f+l\tan\phi} \quad \text{......................... (210.}$$

We then have for any section for which $\lambda_2<\frac{l}{2}$,

$$\left.\begin{array}{rl} S_{max}= & \frac{p}{2l}[(l-x)^2-(l-\lambda_2)^2]\cos\phi-\frac{p}{4f}(\lambda_2^2-x^2)\sin\phi \\ = & p\cdot\cos\phi\cdot\frac{(\lambda_2-x)^2}{2\lambda_2} \end{array}\right\} \text{(211}$$

and

$$\left.\begin{aligned} S_{\min} &= \frac{1}{2} p\,(l - 2\,x)\,\cos\phi - \frac{p\,l^2}{8f}\cdot \sin\phi - S_{\max} \\ &= p\,.\cos\phi\,.\,\frac{3\,\lambda_2 l - 2\,\lambda_2^2 - 2x^2 - l^2}{4\,\lambda_2} \end{aligned}\right\}\ \ldots\ (212.$$

For all sections for which equ. (210) gives $\lambda_2 > \frac{l}{2}$ we must substitute $\lambda_2 = l$ in equ. (211).

If the axis of the arch is of parabolic form, equs. (110) also yield the values of $S.\sec\phi$.

Fig. 50.

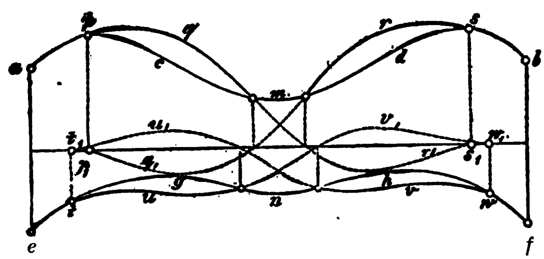
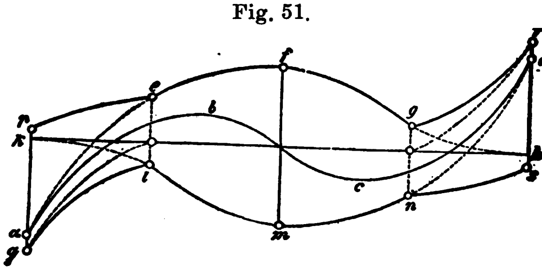


Fig. 51.



Figs. (50) and (51) show the plotted graphs of the maximum moments and shears in a segmental arch having a constant distance between core-points. The curves $a\,c\,m\,d\,b$ and $e\,g\,n\,h\,f$ in Fig. 50 represent the moments about the upper and lower core-points when the entire span is loaded; the curves $a\,p\,q\,r\,s\,b$ and $p_1\,q_1, r_1\,s_1$ give the maximum negative and positive moments about the upper core-points, and the curves $t_1\,u_1, v_1\,w_1$ and $e\,t\,u\,n\,v\,w\,f$ give the same moments about the lower core-points. In Fig. 51, $a\,b\,c\,d$ is the curve of shears for a full-span load, $a\,e\,f\,g\,h$ and $k\,l\,m\,n\,d$ are the curves for loads extending from the section to the right or left ends of the span, respectively, while $p\,e$ or $q\,l$ represents the effect of a load extending from the point J (see Fig. 49) to the end-point B.

If the half-span is loaded, and if the arch-axis is a parabola, the moments in the loaded or unloaded segments, respectively, will be $M = \pm\,\frac{1}{4}\,px\left(\frac{l}{2} - x\right)$, and at the quarter-points $M = \pm\,\frac{1}{64}\,p\,l^2$.

§ 16. Deformations.

1. Effect of Temperature Variation and a Horizontal Displacement of the Abutments. We consider a three-hinged arch whose ends are at the same level. Let the span l be increased by the amount Δl through a yielding of the abutments; also let there be a simultaneous temperature variation of $\pm t°$. Setting $M = 0$ and $P = 0$ in the general deformation equation (185), extending the integration over the half-arch, denoting the total length of axis by b and the length from the end to the point (x, y) by s, and assuming a segmental form of axis for the arch, we find:

$$\frac{\Delta l}{2} = -\Delta \phi_0 \cdot f \pm \omega t \left[\frac{b-l}{2} + \frac{l}{2}\right],$$

hence

$$\Delta \phi_0 = -\frac{\Delta l}{2f} \pm \omega t \frac{b}{2f} \quad \ldots\ldots\ldots\ldots \quad (213.$$

which formula may also be applied to non-segmental arches of flat pitch.

Substituting the expression (213) in (185) and (186), we obtain the following values for the displacements of any point of the arch-axis:

$$\left.\begin{aligned} \Delta x &= -\frac{\Delta l}{2}\left(1 - \frac{y}{f}\right) \pm \omega t \left(s \, . \cos \phi - \frac{b}{2f} \cdot y\right) \\ \Delta y &= -\frac{\Delta l}{2} \cdot \frac{x}{f} \pm \omega t \left(s \, . \sin \phi + \frac{b}{2f} \, . \, x\right. \end{aligned}\right\} \ldots (214.$$

or approximately,

$$\left.\begin{aligned} \Delta x &= -\frac{\Delta l}{2}\left(1 - \frac{y}{f}\right) \pm \omega t \left(x - \frac{ly}{2f}\right) \\ \Delta y &= -\frac{\Delta l}{2} \cdot \frac{x}{f} \pm \omega t \left(y + \frac{lx}{2f}\right) \end{aligned}\right\} \ldots\ldots\ldots \quad (214^a.$$

The upward deflection at the crown is given by the second of equs. (214) as

$$\Delta f = -\frac{l}{4f} \cdot \Delta l \pm \omega t \cdot \frac{bl}{4f} \quad \ldots\ldots\ldots\ldots \quad (215.$$

2. Deformations Due to Loading. In equs. (185) and (186), which serve to determine the deformations, there remains to be determined the quantity $\Delta \phi_0$, i. e., the rotation at the left

end A. The following values may be established for this quantity in three-hinged arches:

If b is the total length of axis, and $\Delta \phi'_c$ the rotation of the right segment of the arch at the crown-section, then, applying equs. (185) and (186) to the two half-arches with the admissible approximations $J=I$ and $P'=P$, we obtain the following expressions for the displacement of the crown:

$$\left.\begin{aligned}
\Delta x_c &= -\Delta \phi_0 \cdot f - \int_0^{\frac{b}{2}} \frac{M}{EI} \cdot (f-y)\, ds \\
&\qquad - \int_0^{\frac{b}{2}} \frac{P}{EA} \left(\frac{f-y}{r} \cdot ds + dx \right) \\
-\Delta x_c &= +\Delta \phi'_c \cdot f + \int_{\frac{b}{2}}^{b} \frac{M}{EI} (y)\, ds \\
&\qquad + \int_{\frac{b}{2}}^{b} \frac{P}{EA} \left(\frac{y}{r}\, ds - dx \right)
\end{aligned}\right\} \quad \ldots\ldots (216.$$

$$\left.\begin{aligned}
\Delta y_c &= \Delta \phi_0 \cdot \frac{l}{2} + \int_0^{\frac{b}{2}} \frac{M}{EI} \left(\frac{l}{2} - x \right) ds \\
&\qquad + \int_0^{\frac{b}{2}} \frac{P}{EA} \left(\frac{l-2x}{2r} \cdot ds - dy \right) \\
-\Delta y_c &= \Delta \phi'_c \cdot \frac{l}{2} + \int_{\frac{b}{2}}^{b} \frac{M}{EI} (l-x)\, ds \\
&\qquad + \int_{\frac{b}{2}}^{b} \frac{P}{EA} \left(\frac{l-x}{r}\, ds - dy \right)
\end{aligned}\right\} \quad \ldots\ldots (217.$$

Adding the first and second of the above equations, also the third and fourth, and eliminating $\Delta \phi'_c$ from the resulting equations, we finally obtain

$$\left.\begin{aligned}
\Delta \phi_0 &= -\int_0^{b} \frac{M}{EI} \left(1 - \frac{y}{2f} - \frac{x}{l} \right) ds \\
&\quad - \int_0^{b} \frac{P}{EA} \left[\left(1 - \frac{y}{2f} - \frac{x}{l} \right) \frac{ds}{r} + \frac{dx}{2f} - \frac{dy}{l} \right]
\end{aligned}\right\} \quad \ldots\ldots (218.$$

Since M and P may now be considered as known for any given loading, the above equation will, in general, enable $\Delta\phi_0$ to be determined and hence, by equs. (185) and (186), we may find the deflections at any point of the arch and, by equs. (216) and (217), the deflections at the crown.

a.) *Flat Parabolic Arch, Loaded with a Concentration.* The equation of the axis of the rib is $y = \frac{4f}{l^2}x(l-x)$; also, for a flat arch, we may put $ds = dx$, $r = \frac{l^2}{8f}$, and P, constant for all sections, $= H$. I and A will likewise be assumed constant. We then obtain for a concentration G, lying at a distance ξ $\left(<\frac{l}{2}\right)$ from the left end, by equ. (218),

$$\Delta\phi_0 = -\frac{G}{EI}\left[\frac{l-\xi}{l}\int_0^\xi x\left(1-\frac{3x}{l}+2\frac{x^2}{l^2}\right)dx+\frac{\xi}{l}\int_\xi^l (l-x)\right.$$

$$\left.\left(1-\frac{3x}{l}+\frac{2x^2}{l^2}\right)dx-\frac{2\xi}{l^2}\int_0^l x(l-x)\left(1-\frac{3x}{l}+\frac{2x^2}{l^2}\right)dx\right]$$

$$-\frac{G}{EA}\frac{\xi}{2f}\left[\frac{8f}{l^2}\int_0^l\left(1-\frac{3x}{l}+\frac{2x^2}{l^2}\right)dx+\int_0^l\left(\frac{dx}{2f}-\frac{dy}{l}\right)\right].$$

Hence,

$$\Delta\phi_0 = \frac{G\,.\,\xi\,[l^3-5\,(l-\xi)^3]}{30\,EI\,.\,l^2}-\frac{G\,.\,\xi\,(8f^2+3\,l^2)}{12\,EAf^2l} \quad \ldots (219.$$

Similarly we find the rotation at the right end B,

$$-\Delta\phi_1 = \frac{G\,.\,\xi\,[l^3-5\,\xi^2\,(l-\xi)]}{30\,EIl^2}-\frac{G\,.\,\xi\,(8f^2+3\,l^2)}{12\,EAf^2l} \quad \ldots (220.$$

The components of the deflection at any point (x, y) of the axis of the rib are found by equs. (185) and (186) to have the following values:

For $x < \frac{l}{2}$ and $< \xi$,

$$\left.\begin{aligned}\Delta x = \frac{G}{EI}\cdot\frac{f\,.\,x}{15\,l^4}\Big[\,&2\,\xi\left\{5\,(l-\xi)^3-l^3\right\}(l-x)\\ &-\left\{5\,l^2(2l-3x)-\xi\,(30\,l^2-55\,lx+16\,x^2)\right\}x^2\Big]\\ &+\frac{G}{EA}\frac{\xi x}{6fl^4}\left\{16\,(l-2x)\,f^2+3\,l^3\right\}(l-2x)\end{aligned}\right\} \quad \ldots (221.$$

$$\left.\begin{aligned}\Delta y=&\frac{G}{EI}\cdot\frac{x}{30l^2}\Big[5x^2\Big\{l^2-\xi\,(3l-x)\Big\}\\&\qquad+\xi\Big\{l^3-5\,(l-\xi)^3\Big\}\Big]\\&+\frac{G}{EA}\cdot\frac{\xi x}{12f^2l^2}\,[16\,(3x-2l)\,f^2-3l^3]\end{aligned}\right\}\ldots(222.$$

For $x<\frac{l}{2}$ but $>\xi$:

$$\left.\begin{aligned}\Delta x=&\frac{G}{EI}\,\frac{f\xi}{15\,l^4}\,[-5\xi^2l^2(2l-\xi)+2xl\,(4l^3+15l\xi^2-5\xi^3)\\&-2x^2(19l^3+15\xi^2l-5\xi^3)+70l^2x^3-55lx^4+16x^5]\\&+\frac{G}{EA}\,\frac{\xi x}{6fl^4}\,[16\,(l-2x)\,f^2+3l^3]\,(l-2x)\end{aligned}\right\}(223.$$

$$\left.\begin{aligned}\Delta y=&\frac{G}{EI}\cdot\frac{\xi}{30\,l^2}\,[5\xi^2l^2-x\,(4l^3+15l\xi^2-5\xi^3)+15x^2l^2\\&-15x^3l+5x^4]+\frac{G}{EA}\cdot\frac{\xi.x}{12\,f^2l^2}\,[16(3x-2l)\,f^2-3l^3]\end{aligned}\right\}(224.$$

Finally, for $x>\frac{l}{2}$ (and hence $>\xi$):

$$\left.\begin{aligned}\Delta x=&\frac{G}{EI}\cdot\frac{f.\xi}{15\,l^4}\,[-l^5+2xl(6l^3-5\xi^2l+5\xi^3)\\&-2x^2\;(21l^3-5\xi^2l+5\xi^3)\;+70\,x^3l^2-55\,x^4l\;+16\,x^5\Big]\\&-\frac{G}{EA}\,\frac{\xi(l-x)}{6fl^4}\Big\{16\;(2x-l)\;f^2+3\,l^3\Big\}(2x-l).\end{aligned}\right\}(225.$$

$$\left.\begin{aligned}\Delta y=&\frac{G\xi\,(l-x)}{30\,EIl^2}\,[l^3-5\xi^2(l-\xi)-5x\,(l-x)^2]\\&\qquad+\frac{G\xi(l-x)}{12\,EAf^2l^2}\,[16(l-3x)f^2-3l^3]\end{aligned}\right\}\ldots\ldots(226.$$

The expressions for the vertical deflections agree in the first parts, viz., the terms depending on the moments, with the corresponding terms given in equs. (139) to (141) for the system of arches or cables stiffened with a three-hinged truss. We may therefore apply here also the graphic construction there given for determining the deflections.

The *displacements at the crown* are found to be:

$$\Delta x_c=\frac{Gf\xi}{6\,EIl^2}\Big(\xi^3+\frac{1}{8}\,l^3-\xi^2l\Big)\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\;(227.$$

$$\Delta y_c=\frac{G.\xi}{60\,EIl}\Big(5\xi^3+\frac{3}{8}\,l^3-5\xi^2l\Big)-\frac{G\xi(8f^2+3l^2)}{24\,EAf^2}\ldots\ldots(228.$$

The rotations at the crown are found to be, for the left arch-segment,

$$\Delta\phi_c=\frac{G\xi}{120.\,EIl^2}(9l^3-60\xi^2l+20\xi^3)-\frac{G\xi}{12\,EAf^2l}(3l^2-16f^2)\ldots(229.$$

and for the right arch-segment,

$$\Delta\phi'_c = \frac{G\xi}{120 . EI\,l^2}(l^3+20\xi^2 l-20\xi^3)+\frac{G\xi}{12 EA f^2 l}(3l^2-16f^2) \quad (230.$$

The preceding formulae apply only if $\xi < \frac{l}{2}$; the effect of a load in the right half of the arch is obtainable by the principle of symmetry.

If the load is at the crown, there results

$$\Delta x_c = 0$$

$$\Delta y_c = -\frac{Gl^3}{480\,EI}-\frac{Gl(8f^2+3l^2)}{48\,EA f^2} \quad (231.$$

$$\Delta\phi_c = -\Delta\phi'_c = -\frac{7Gl^2}{480\,EI}+\frac{G(16f^2-3l^2)}{24 EA f^2} \quad (232.$$

By equ. (227), the horizontal displacement of the crown will be a maximum for a load at $\xi = \frac{l}{4}$; furthermore, every load on the left half of the rib produces a displacement of the crown toward the right. Also, if we consider only the first term in equ. (228), we find $\Delta y_c = 0$ at about $\xi = 0.336\,l$, so that, approximately, all loads in the outer thirds of the span produce an uplift, and all loads in the middle third produce a depression of the crown.

b.) *Flat Parabolic Arch; Continuous Load.* The deformations produced by any loading, whether a train of concentrations or a continuous load, are readily evaluated by means of the influence lines which may be constructed in accordance with equs. (219) to (230). Let us consider here only the case of a uniformly distributed load of intensity p extending for a distance λ from the left end, where $\lambda < \frac{l}{2}$. For such loading we obtain the following formulae:

$$\Delta\phi_0 = -\frac{p\lambda^2}{120\,EI\,l^2}[8l^3-20l^2\lambda+15l\lambda^2-4\lambda^3]-\frac{p\lambda^2(8f^2+3l^2)}{24\,EA f^2 l} \quad (233.$$

$$-\Delta\phi_1 = \frac{p\lambda^2}{120\,EI\,l^2}[2l^3-5l\lambda^2+4\lambda^3]-\frac{p\lambda^2(8f^2+3l^2)}{24 EA f^2 l} \quad (234.$$

and, at the crown,

$$\Delta\phi_c = \frac{p\lambda^2}{240\,EI\,l^2}[9l^3-30l\lambda^2+8\lambda^3]-\frac{p\lambda^2(3l^2-16f^2)}{24\,EA f^2\,l} \quad (235.$$

$$\Delta\phi'_c = \frac{p\lambda^2}{240\,EI\,l^2}[l^3+10l\lambda^2-8\lambda^3]+\frac{p\lambda^2(3l^2-16f^2)}{24 EA f^2 l} \quad (236.$$

$$\Delta x_c = \frac{p\lambda^2 f}{480\,EI\,l^2}[5l^3-20\,l\lambda^2+16\lambda^3] \quad (237.$$

$$\Delta y_c = \frac{p\lambda^2}{960\,EI\,l}[3l^3 - 20\,l\,\lambda^2 + 16\lambda^3] - \frac{p\lambda^2(8f^2+3\,l^2)}{48\,EA\,f^2} \quad \ldots\ldots\ (238.$$

If the loaded length λ extends from the left end to some point beyond the crown, i. e., if $\lambda > \frac{l}{2}$, we have the following formulae:

$$\left.\begin{aligned}\Delta\phi_0 = &-\frac{p(l-\lambda)^2}{120\,EI\,l^2}[2l^3 - 5l(l-\lambda)^2 + 4(l-\lambda)^3]\\ &-\frac{p(4l\lambda - l^2 - 2\lambda^2)(8f^2+3\,l^2)}{48\,EA\,f^2\,l}\end{aligned}\right\}\ \ldots\ (239.$$

$$\left.\begin{aligned}-\Delta\phi_1 = &\frac{p(l-\lambda)^2}{120\,EI\,l^2}[-l^3 + 2\,l^2\lambda + 3l\,\lambda^2 + 4\,\lambda^3]\\ &-\frac{p(4l\lambda - l^2 - 2\lambda^2)(8f^2+3\,l^2)}{48\,EA\,f^2\,l}\end{aligned}\right\}\ \ldots\ (240.$$

$$\left.\begin{aligned}\Delta\phi_c = &\frac{p(l-\lambda)^2}{240\,EI\,l^2}[l^3 + 10l(l-\lambda)^2 - 8\,(l-\lambda)^3]\\ &-\frac{p(4l\lambda - l^2 - 2\lambda^2)(3\,l^2-16f^2)}{48\,EA\,f^2 l}\end{aligned}\right\}\ \ldots\ (241.$$

$$\left.\begin{aligned}\Delta\phi'_c = &\frac{p(l-\lambda)^2}{240\,EI\,l^2}[9l^3 - 30l(l-\lambda)^2 + 8(l-\lambda)^3]\\ &+\frac{p(4l\lambda - l^2 - 2\lambda^2)(3\,l^2-16f^2)}{48\,EA\,f^2\,l}\end{aligned}\right\}\ \ldots\ (242.$$

$$\Delta x_c = \frac{p(l-\lambda)^2 f}{480\,EI\,l^2}[5l^3 - 20l(l-\lambda)^2 + 16\,(l-\lambda)^3] \quad \ldots\ldots\ (243.$$

$$\left.\begin{aligned}\Delta y_c = &\frac{p(l-\lambda)^2}{960\,EI\,l}[3\,l^3 - 20l(l-\lambda)^2 + 16\,(l-\lambda)^3]\\ &-\frac{p(4l\lambda - l^2 - 2\lambda^2)(3\,l^2+8f^2)}{96\,EA\,f^2}\end{aligned}\right\}\ \ldots\ (244.$$

The horizontal deflection of the crown is found to be a maximum for $\lambda = \frac{l}{2}$; we then obtain

$$\max.(\Delta x_c) = \frac{p\,f\,l^3}{960\,EI} \quad \ldots\ldots\ldots\ldots\ (245.$$

The maximum uplift of the crown occurs approximately when the first and last thirds of the span are loaded; we then obtain

$$\max.(\Delta y_c) = \frac{37}{116{,}640}\frac{pl^4}{EI} - \frac{pl^2(3\,l^2+8f^2)}{216\,EA\,f^2} \quad \ldots\ (246.$$

The maximum downward deflection at the crown, which occurs approximately when the middle third of the span is loaded, is found to be

$$\max.(-\Delta y_c) = \frac{37}{116{,}640}\frac{pl^4}{EI} + \frac{5}{864}\frac{pl^2(3\,l^2+8f^2)}{EA\,f^2} \quad \ldots\ldots\ldots\ (247.$$

2. ARCHED RIB WITH END HINGES.

§ 17. Determination of the Horizontal Thrust.

1. General Case. By equations (187) to (192), putting both of the end moments M_1 and M_2 equal to zero, the external forces may be determined as soon as the horizontal force H is known. To determine the latter it is necessary to consider the elastic deformations. The appropriate equation of condition may be developed from the Principle of Virtual Work, or it may be derived directly from equ. (185). From the latter, with the approximation $J = I$, we obtain the displacement of the end B toward A:

$$\Delta l = \int_0^b \frac{M}{E\,I}\,y\,d\,s + \int_0^l \frac{P'}{E\,A}\left(\frac{y}{r}\,ds - dx\right) - \omega t \int_0^l \left(\frac{y\,d\,s}{r} - dx\right) \qquad (248.$$

By the Principle of Virtual Work, equ. (248) may be established as follows: The general expression for the work of deformation of curved ribs may be written,

$$W = \int \frac{P'^2\,d\,s}{2\,E\,A} + \int \frac{M^2 ds}{2\,E\,J} - \int \omega\,t\,P' \,.\, d\,s,$$

where, with the previous notation, $P' = P + \frac{M}{r}$, and, for the two-hinged arch, $M = \mathbf{M} - Hy$ and $P' = V \sin\phi + H \cos\phi + \frac{\mathbf{M}}{r} - \frac{H\,y}{r}$. But, by the above mentioned principle, $\frac{d\,W}{d\,H}$ must equal the virtual work per unit value of the horizontal end-forces H; i. e., $\frac{d\,W}{d\,H} = -\Delta\,l$ if the two ends increase their horizontal distance from each other by $\Delta\,l$. (cf. Castigliano's Theorem). We thus obtain

$$-\Delta l = \int \frac{P'}{E\,A} \cdot \frac{d\,P'}{d\,H} \cdot d\,s + \int \frac{M}{E\,J} \cdot \frac{d\,M}{d\,H} \cdot d\,s - \int \omega\,t\,\frac{d\,P'}{d\,H} \,.\, d\,s$$

or

$$\Delta\,l = \int \frac{P'}{E\,A}\left(\frac{y}{r} - \cos\phi\right) d\,s + \int \frac{M}{E\,J}\,y \cdot d\,s - \omega\,t \int \left(\frac{y}{r} - \cos\phi\right) d\,s,$$

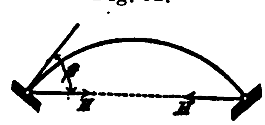
which, as $ds \cos\phi = d\,x$, is identical with equ. (248).

In flat arches $\left(\text{up to about } \frac{f}{l} = \frac{1}{5}\right)$ we have, very nearly,

$$P' = H \,.\, \frac{d\,s}{d\,x} = H \,.\, \sec\phi$$

and, if the curvature is approximately constant, we may write

Fig. 52.



$$\frac{y}{r} d s - d x = -d s \cdot \cos \beta;$$

this is an approximation always admissible in practical applications on account of the relatively small effect of P upon the end deflections.

Introducing also an average cross-section A_0, defined by the formula

$$\frac{b}{A_0} = \int_0^b \frac{ds}{A \cos \phi} = \frac{s_1}{A_1 \cos \phi_1} + \frac{s_2}{A_2 \cos \phi_2} + \dots \quad (249.$$

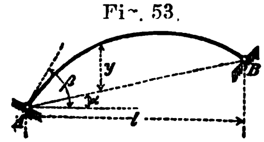
where b is the total length of axis of the arch, $A_1 A_2$..the mean cross-sections of the arch elements, $s_1 s_2$..their lengths, and $\phi_1 \phi_2$.. their inclinations to the horizontal, and substituting $M = \mathbf{M} - H y$, equ. (248) yields the following expression for the horizontal thrust:

$$H = \frac{\int_0^b \frac{\mathbf{M} y}{I} \cdot ds + E\omega t \cdot b \cdot \cos \beta - E \Delta l}{\int_0^b \frac{y^2 ds}{I} + \frac{b \cos \beta}{A_0}} \quad \dots\dots (250.$$

It should here be observed that the abutment points of the arch need not lie at the centers of gravity of the end sections, but may be located either higher or lower at will. It is merely necessary to always measure y from the line joining the end points.

If the two ends are not at the same elevation, but have their connecting line forming an angle α with the horizontal (Fig. 53), then the horizontal component of the end reactions is given by the following expressions:

Fig. 53.



a.) In case the span is increased by the amount Δl through a horizontal displacement of the abutments, A, B,

$$H = \frac{\int_0^b \frac{\mathbf{M} y}{I} \cdot ds + E \omega t \cdot b \cdot \cos (\beta - \alpha) \cdot \sec \alpha - E \cdot \Delta l}{\int_0^b \frac{y^2 ds}{I} + \frac{b . \cos (\beta - \alpha) . \sec^2 \alpha}{A_0}} \quad \dots (251^a.$$

b.) In case the abutments are displaced in the direction of the connecting line $A\,B$ through the distance $\Delta\, l. sec\ \alpha$,

$$H = \frac{\int_0^b \frac{\mathbf{M}\, y}{I} . ds + E\,\omega\, t . b \cos\,(\beta - \alpha) . \sec \alpha - E\,\Delta\, l \sec^2 \alpha}{\int_0^b \frac{y^2\, d\, s}{I} + \frac{b . \cos\,(\beta - \alpha) . \sec^2 \alpha}{A_0}} \quad \ldots (251^b.$$

In flat arches we may also write, with close approximation, $b \cos \beta = l$ and $b \cos (\beta - \alpha) = l . \sec\ \alpha$.

In the above expressions for H, the first term in the numerator gives the effect of the loading, the second term that of a variation of temperature, and the third term that of a displacement of the abutments. The symbol t denotes the difference of temperature in the arch from that of erection, i. e., from the temperature at which the arch, supposed to be without load or weight, would be entirely unstressed. In a full-centered (semicircular) arch, for which the above formulae are by no means sufficiently accurate, no stresses are producible by changes of temperature.

If the moment of inertia of the cross-section of the arch is constant within each panel-length $a_1\, a_2 \ldots a_m \ldots$, then the definite integrals in the preceding expressions for H may be transformed into summations exactly as developed in § 5. For this purpose we must introduce the modified moments of inertia,

$$I' = I \cos \phi = I \frac{dx}{ds} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (252.$$

We then obtain, on the basis of equs. (72) and (73), if a_0 is a mean panel-length and I'_0 a mean value of I',

$$\int_0^b \frac{\mathbf{M}\, y}{I} . ds = \frac{a_0}{I'_0} \sum_0^l \mathbf{M}_m\, v_m, \ldots\ldots\ldots\ldots\ldots (253.$$

$$\int_0^b \frac{y^2}{I} . ds = \frac{a_0}{I'_0} \sum_0^l y_m\, v_m, \ldots\ldots\ldots\ldots\ldots (254.$$

where

$$v_m = \frac{a_m}{6 a_0} \cdot \frac{I'_0}{I'_m} \left(2\, y_m + y_{m-1}\right) + \frac{a_{m+1}}{6\, a_0} \cdot \frac{I'_0}{I'_{m+1}} \left(2\, y_m + y_{m+1}\right) \ldots (255.$$

Thus the same rules apply for an arch carrying a vertical concentration, as have been established for the suspension systems considered in §§ 5 *et seq.*; the horizontal thrust produced by the action of such load may be taken proportional to the

ordinates of a funicular polygon constructed by treating the quantities v as forces applied at the respective panel-points.

If the loading consists only of a *horizontal* force W, acting at a height h above the closing chord at a point of the arch-axis whose abscissa is w, then, if we omit the effects of temperature variation and of shifting of the abutments and substitute for the axial force the approximate expressions

$$P=(H-W)\sec\phi\Big]_{x=0}^{x=w} \quad \text{and} \quad P=H\sec\phi\Big]_{x=w}^{x=l},$$

the fundamental equation (248) yields the following formula for the horizontal thrust at the right abutment:

$$H_w=\frac{\int_0^b \frac{My}{I}\,.\,ds+W\,.\,\cos\beta\int_0^w\frac{ds}{A\,.\cos\phi}}{\int_0^b\frac{y^2 ds}{I}+\frac{b\,.\cos\beta}{A_0}} \quad \ldots\ldots (256.$$

or, substituting the approximate value $\frac{W\,.\,w}{A_0}$ for the second term of the numerator, and replacing the definite integrals by the summation expressions of (253) and (254),

$$H_w=\frac{\sum_0^l \mathbf{M}_m v_m+W\,.\,\frac{I'_0\,.\,w}{A_0\,.\,a_0}}{\sum_0^l y_m v_m+\frac{b\,.\cos\beta\,.\,I'_0}{A_0 . a_0}} \quad \ldots\ldots\ldots\ldots (257.$$

At the left end, there is acting, then, the outward force $W-H_w$.

But, for the single horizontal load W, we have

$$\mathbf{M}=W\cdot y-W\,.\,\frac{h}{l}\,.\,x\Big]_{x=0}^{x=w} \quad \text{and} \quad \mathbf{M}=W.\,\frac{h}{l}\,.\,(l-x)\Big]_{x=w}^{x=l};$$

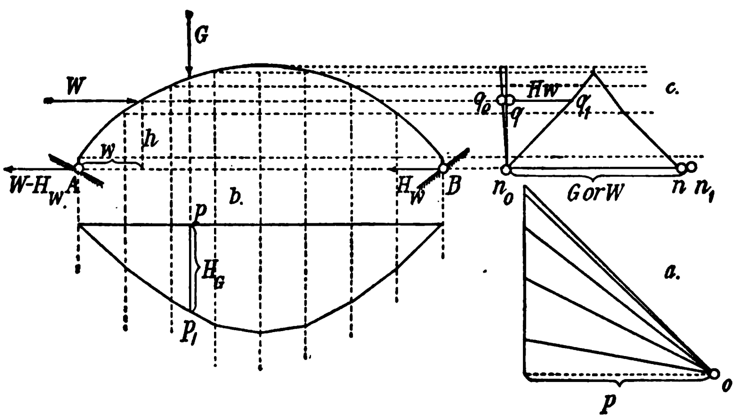
hence

$$\left.\begin{aligned}\sum_0^l \mathbf{M}_m v_m&=W\left[\sum_0^w y_m v_m-\frac{h}{l}\sum_0^w v_m\,.\,x+\frac{h}{l}\sum_w^l v_m(l-x)\right]\\&=W\left[\frac{h}{l}\sum_0^l(l-x)v_m-\sum_0^w(h-y)\,v_m\right]\end{aligned}\right\}\ldots (258.$$

The expression $\frac{1}{l}\sum_0^l (l-x)\,v_m$ gives the reaction at A for the forces v_m considered as acting vertically. If we imagine this reaction, which is easily determined and $=\frac{1}{2}\sum_0^l v_m$ in sym-

metrical arches, to be applied horizontally at A, also the forces v_m to be acting in the opposite direction at the respective panel-points, then the above expression in brackets denotes the static moment of those forces about the line of action of W and may be represented by the ordinates of a funicular polygon in the well-known manner.

Fig. 54.



The construction for the horizontal thrust produced by a vertical load G, also by a horizontal force W, is shown in Fig. 54. Fig. 54[a] is the force polygon constructed with the quantities v_m computed by equ. (255); Fig. 54[b] is the funicular polygon of these forces considered as acting vertically; Fig. 54[c] is the funicular polygon for the same forces acting horizontally. These funicular polygons give the intercepts pp_1 and qq_1 on the lines of action of G and W respectively. If we add to the latter intercept the small distance $qq_0 = \frac{I'_0 w}{A_0 . a_0 . p}$, then the length pp_1 and $q_0 q_1$ will represent the horizontal thrusts for the forces G and W respectively, provided the latter forces are represented by the distance $n_0 n_1$. The latter length is composed of

$$n_0 n = \sum_0^l y_m v_m \quad \text{and} \quad n n_1 = \frac{b . \cos\beta . I'_0}{A_0 . a_0 . p}$$

where p is the pole distance of the force polygon (Fig. 54[a]).*

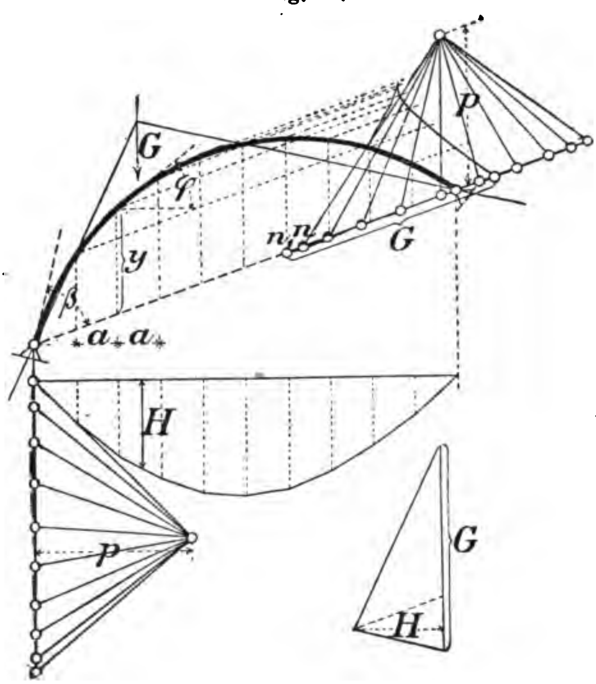
With a_m and I' constant, i. e., with the moment of inertia increasing toward the abutments in the same ratio as $\sec\phi$, we obtain approximately $v_m = y_m$.

* That the H-curve for concentrated loads on arches with end hinges may be represented by a funicular polygon, was first demonstrated by *Mohr*.

In plate-girder arches, in which the depth of web is usually fairly constant, the approximation $v_m = y_m$ will always give sufficiently accurate results. It appears that the variation in the moments of inertia must be very marked before it will exert any appreciable influence upon the value of H.

If the two ends lie in a line inclined at an angle α to the horizontal, the preceding method for determining the horizontal thrust H must be modified as shown in Fig. 55. The quantities

Fig. 55.



v are given here, as above, by equ. (255), the ordinates y being measured from the closing chord; and, in $I' = I \cos \phi$, ϕ denotes the inclination of the arch elements to the horizontal. The funicular polygon giving the quantity $\Sigma\, yv$ must be constructed for the forces v considered as acting parallel to the arch-chord, using a force polygon whose vertical pole distance is p. The correction length is, in this case, $nn_1 = \dfrac{b \,.\, \cos\beta \,.\, \sec^2\alpha \,.\, I'_0}{A_0 \,.\, a_0 \,.\, p}$.

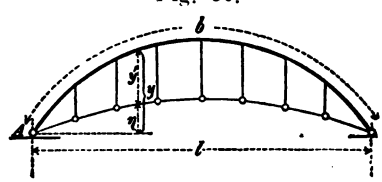
Equ. (250), likewise (251^a) and (251^b), may be extended to the case of an arch having its two ends connected by an elastic tie. If A_1 is the cross-section of this tie-rod, the lengthening of

the span becomes $\Delta l = \frac{H}{E A_1} . l + \omega t l$, so that equ. (250) assumes the form:

$$H = \frac{\int_0^b \frac{\mathbf{M} y}{I} . ds + E \omega t . (b \cos \beta - l)}{\int_0^b \frac{y^2}{I} . ds + \frac{b . \cos \beta}{A_0} + \frac{l}{A_1}} \quad \ldots\ldots (259.$$

The problem becomes somewhat more difficult if the tie-rod itself is arched and is connected to the main arch by suspension rods. The equation for H may then be established by applying the Principle of Least Work. If the suspension rods are distributed at uniform intervals of length a (Fig. 56), and accepting the approximation, admissible in flat arches, of equating the axial forces in the arch to H, we have

Fig. 56.



$$M = \mathbf{M} - H y + H \eta = \mathbf{M} - H y';$$

and if

A_0 = a mean cross-section of the rib,
A_1 = the cross-section of the tie-rod,
A_2 = the cross-section of a suspension rod,
f_1 = the rise of the tie-rod,

and $\quad l_1 = l\left(1 + \frac{16}{3}\frac{f_1^2}{l^2}\right)$,

then we obtain,

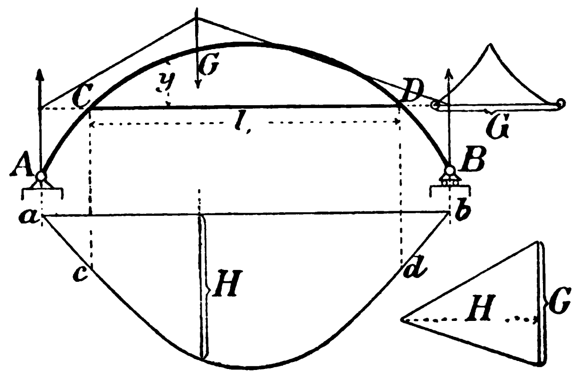
$$H = \frac{\int_0^b \frac{\mathbf{M} y'}{I} ds + E \omega t (b \cos \beta - l_1)}{\int_0^b \frac{y'^2}{I} \cdot ds + \frac{b \cos \beta}{A_0} + \frac{l_1}{A_1} + \frac{1}{A_2} \sum_0^l y' \left(\frac{\Delta^2 \eta}{a}\right)^2} \quad \ldots\ldots (260.$$

This formula leads to the same graphic method for determining H as has been deduced above from equ. (250) *et seq.*, but in place of the arch-ordinates y there must be introduced the intercepts y' between arch and tension-rod.

If the tie-rod does not connect the ends of the axis but two higher points thereof ($C D$, Fig. 57), its stress is to be computed by the formula:

$$H = \frac{\int_C^D \frac{\mathbf{M}y}{I} \cdot ds + E\,\omega\,t(b\cos\beta - l_1)}{\int_C^D \frac{y^2}{I} \cdot ds + \frac{b_1 \cos\beta}{A_0} + \frac{l_1}{A_1}}$$

Fig. 57.



Here y is the arch-ordinate measured from the tie-rod, $\mathbf{M}$ is the moment of the given loading in a simple beam of span $AB = l$, l_1 and A_1 are the length and cross-section of the tension-rod, and b_1 and A_0 are the length and mean section of the arch between the points C and D. The influence line for H is given by the funicular polygon of the forces v (computed from the values of y); this polygon is prolonged outward, above the points c and d, to the closing side $a\,b$.

If the arch is *connected through hinges to elastic piers,* and if h is their height, I_1 their constant moment of inertia, and E_1 the coefficient of elasticity of their material, then the outward deflection of each pier by the force H must equal $\frac{\Delta l}{2}$, so that the increase in span will be $\Delta l = \frac{2}{3}\frac{H \cdot h^3}{E_1 I_1}$. Substituting this in equ. (250) and solving for H, we find

$$\left. H = \frac{\int_0^l \frac{\mathbf{M}y}{I} \cdot ds}{\int_0^l \frac{y^2}{I} \cdot ds + \frac{b\cos\beta}{A_0} + \frac{2}{3}\frac{E}{E_1} \cdot \frac{h^3}{I_1}} \right\} \dots (259^a.$$

2. Arch with Flat Parabolic Axis and Parallel Flanges. Concentrated Load. Adopting the same simplifying assump-

tions as above, equ. (250) for determining the horizontal thrust is also applicable to the present case. Assuming $A \cos\phi = A_0$, constant, likewise $I.\cos\phi = I\frac{dx}{ds} = I_0$, constant, and noting that, for flat parabolic arches, $b \cos\beta \doteq l\left(1+\frac{8}{3}\frac{f^2}{l^2}\right)\left(1-8\frac{f^2}{l^2}\right) \doteq l\left(1-\frac{16}{3}\frac{f^2}{l^2}\right)$, then for a parabola of rise f with a load G at a distance ξ from the end, we have (cf. the derivation of equ. 75):

$$\left.\begin{aligned} H = & \frac{5\xi(\xi^3 - 2l\xi^2 + l^3)f}{\left[8f^2 + 15\frac{I_0}{A_0}\left(1-\frac{16}{3}\frac{f^2}{l^2}\right)\right]l^3} \cdot G \\ & + 15EI_0 \frac{\omega t\left(1-\frac{16}{3}\frac{f^2}{l^2}\right) - \frac{\Delta l}{l}}{8f^2 + 15\frac{I_0}{A_0}\left(1-\frac{16}{3}\frac{f^2}{l^2}\right)} \end{aligned}\right\} \qquad (261.$$

For the effect of a concentration alone, we obtain from the first term of the above expression

$$H = \frac{5}{8}\frac{\xi}{l}\left(\frac{\xi^3}{l^3} - 2\frac{\xi^2}{l^2} + 1\right)\frac{l}{f'} \cdot G \qquad (261^a.$$

where, for abbreviation,

$$f' = f\left[1 + \frac{15}{8}\frac{I_0}{A_0 f^2}\left(1 - \frac{16}{3}\frac{f^2}{l^2}\right)\right] \qquad (262.$$

The substitution of $f' = f$ would be an approximation corresponding to the assumption that the axial force contributes but very little to the deformation of the arch.

The equation of the reaction locus is derived from equ. (198^a):

$$\eta = \frac{8f'l^2}{5(l^2 + l\xi - \xi^2)} \qquad (263.$$

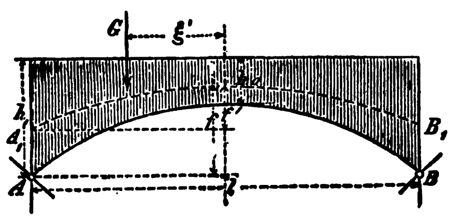
A much simplified, approximate expression for the horizontal thrust in a flat, parabolic, two-hinged arch of uniform section has been established by *Engesser* and *Müller-Breslau*; it reads:

$$H = \frac{3}{4}G \cdot \frac{\xi(l-\xi)}{l.f'} \qquad (263^a.$$

With this approximation, the influence line for H becomes a parabola, and the reaction locus a straight line parallel to the arch-chord. The error in comparison with the more exact equation (261^a) amounts to -4% for a load at the crown, but, in the small values of the horizontal thrust for loads near the abutments, the error becomes relatively greater up to about $+10\%$.

3. Plate-Girder Arch with Parabolic Intrados and Straight Extrados. Although the general formulae [equ. (250) together with (253) to (255)] are sufficient for the treatment of this case, let us establish a direct, approximate formula for the horizontal thrust producible by a concentrated load. If $A_1 B_1$ is the gravity axis of the rib (Fig. 58), f' its rise, f the height

Fig. 58.



of its crown above the line connecting the end hinges, h_0 and h_1 the effective depths (or distances center to center of flanges) of the cross-sections at the crown and abutment respectively, and I_0 and I_1 the moments of inertia of the same cross-sections; also introducing the ratio-symbols

$$\left.\begin{aligned} \nu &= \frac{f'}{f} \\ \epsilon &= \frac{h_1 - h_0}{h_0} \\ \chi &= \frac{I_1 + I_0}{I_0} - \frac{2\,h_1}{h_0} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (264.$$

and the abbreviations

$$\gamma = \frac{2\,\xi'}{l}$$

and, approximately,

$$I'_x = I_0\left(1 + 8\,\epsilon\frac{x^2}{l^2} + 16\,\chi\frac{x^4}{l^4}\right)$$

where x is the distance of any section from the crown; then equ. (250) yields the following formula for the horizontal thrust producible by a load placed at a distance ξ' from the crown:

$$H = \frac{\left\{\frac{1}{8}(1-\gamma^2) + \frac{1}{48}\cdot\mathbf{a}(1-\gamma^4) - \frac{1}{120}\mathbf{b}(1-\gamma^6) + \frac{1}{224}\mathbf{c}(1-\gamma^8)\right\} G\frac{l}{f} + \frac{E I_0}{f^2 l}\,.\omega t\,b\cos\beta}{1 - \frac{2}{3}\nu + \frac{1}{5}\nu^2 + 8\,\nu^2\frac{f^2}{l^2}\left(\frac{1}{3} - \frac{2}{5}\nu + \frac{1}{7}\nu^2\right) - 2\epsilon\,\mathbf{d} + (4\epsilon^2 - \chi)\,\mathbf{e} - \frac{4}{7}\epsilon(2\epsilon^2 - \chi) + \frac{I_0\,\theta\cos\beta}{A_0 f^2 l}} \quad \ldots. \quad (265$$

9

where

$$\mathbf{a} = \left(8\nu\frac{f^2}{l^2} - 1\right)\nu - 2\epsilon$$

$$\mathbf{b} = 8\nu^3\frac{f^2}{l^2} + 2\nu\epsilon\left(8\nu\frac{f^2}{l^2} - 1\right) - 4\epsilon^2 + \chi$$

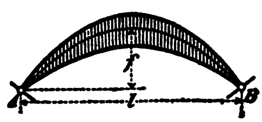
$$\mathbf{c} = 16\epsilon\nu^3\frac{f^2}{l^2} + (4\epsilon^2 - \chi)\,\nu\left(8\nu\frac{f^2}{l^2} - 1\right) - 4\epsilon(2\epsilon^2 - \chi)$$

$$\mathbf{d} = \frac{1}{3} - \frac{2}{5}\nu + \frac{1}{7}\nu^2 + 8\nu^2\frac{f^2}{l^2}\left(\frac{1}{5} - \frac{2}{7}\nu\right)$$

$$\mathbf{e} = \frac{1}{5} + \frac{2}{7}\nu\left(4\nu\frac{f^2}{l^2} - 1\right)$$

The above equation may also be used as an approximate, general formula for calculating the horizontal thrust in any arch-rib having a parabolic gravity-axis and any continuous variation of depth of girder. Thus, if the cross-section is uniform, we may obtain the horizontal thrust from the above formula by putting $\epsilon = \chi = 0$ and, if the end hinges are

Fig. 59.



in the axis, $\nu = 1$. For the crescent arch (Fig. 59), we must write $\nu = 1$, $\epsilon = -1$, $\chi = +1$; we then obtain for this case,

$$H = \frac{\left\{\frac{1}{8}(1-\gamma^2) + \left[\frac{1}{48}(1-\gamma^4) + \frac{1}{120}(1-\gamma^6) + \frac{1}{224}(1-\gamma^8)\right]\left(1+\frac{8f^2}{l^2}\right)\right\}G.\frac{l}{f} + \frac{EI_0}{f^2 l}.\omega\, t\, b\cos\beta}{1+\frac{8}{3}\frac{f^2}{l^2}+\frac{I_0 b\cos\beta}{A_0 f^2 l}} \quad \ldots\ldots (266.$$

Another formula for crescent arches, given by *Schäffer*, is as follows:

$$H = \frac{\left\{(1+\gamma)\log_e\frac{2}{1+\gamma} + (1-\gamma)\log_e\frac{2}{1-\gamma}\right\}G\cdot\frac{l}{8f} + \frac{EI_0}{f^2 l}\cdot\omega\, t\, b\cos\beta}{1+\frac{8}{3}\frac{f^2}{l^2}+\frac{I_0 b\cos\beta}{A_0 f^2 l}} \quad \ldots (266^a$$

The reaction locus for the above types of arches is determined by the general equation (198^a).

4. Segmental Arch of Uniform Cross-Section. Substituting

$$P' = P + \frac{M}{r},$$

$$y = r(\cos\phi - \cos\beta), \qquad y_0 = y_1 = 0, \qquad \frac{y}{r}ds - dx = -ds\cos\beta$$

$$\text{and} \quad P = H\cos\phi + V\sin\phi$$

also putting the small quantity

$$\frac{I}{A r^2} = \delta \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (267.$$

where I is supposed to be constant, equ. (185) yields the following relation:

$$\left.\begin{aligned}
E\,I\,.\,\Delta\,l = r^2 \int_{-\beta}^{+\beta} \mathbf{M}\,(\cos\phi - \cos\beta)\,d\phi - r^2\,\delta \int_{-\beta}^{+\beta} \mathbf{M}\cos\beta\,d\phi \\
-2\,H\,r^3 \int_0^{\beta} (\cos\phi - \cos\beta)^2 d\phi + 2 H r^3 \delta \int_0^{\beta} (\cos\phi - \cos\beta)\cos\beta\,d\phi \\
-2\,H\,r^3\,\delta \int_0^{\beta} \cos\beta . \cos\phi . d\phi - \delta r^3 \int_{-\beta}^{+\beta} V \cos\beta . \sin\phi . d\phi \\
+2\,E\,I\,\omega\,t\cos\beta\,.\,r\,.\,\beta
\end{aligned}\right\} \quad (268.$$

If two concentrations G are applied at two symmetrically located points of the arch, and if the included central angle $= 2\,\gamma$, we find

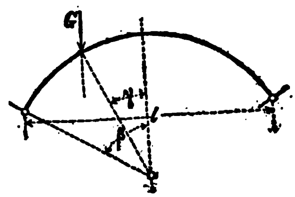


Fig. 60.

$$\mathbf{M} = G\,r\,(\sin\beta - \sin\gamma)\Big]_{\phi=0}^{\phi=\gamma} \qquad V = 0\Big]_{\phi=0}^{\phi=\gamma}$$

$$\mathbf{M} = G\,r\,(\sin\beta - \sin\phi)\Big]_{\phi=\gamma}^{\phi=\beta} \qquad V = G\Big]_{\phi=\gamma}^{\phi=\beta}$$

Substituting these expressions in equ. (268) and solving for H, we derive the following formula for the horizontal thrust produced by a single concentration (together with a temperature change t and an end-deflection $\Delta\,l$):

$$H = \frac{[\sin^2\beta - \sin^2\gamma + 2\cos\beta\,(\cos\gamma - \cos\beta) - 2(1+\delta)\cos\beta\,(\beta\sin\beta - \gamma\sin\gamma)\,]\,G + 2\frac{E\,I}{r^3}\,(2\,\omega\,t\,r\,\beta\cos\beta - \Delta\,l)}{2[\beta - 3\sin\beta\cos\beta + 2\,(1+\delta)\beta\,.\cos^2\beta]} \quad \ldots\ldots (269.$$

In the semicircular arch, $\beta = \frac{\pi}{2}$; consequently,

$$H = \frac{\cos^2\gamma}{\pi}\,G - \frac{2\,E\,I\,\Delta\,l}{r^3\,\pi} \quad \ldots\ldots\ldots\ldots\ldots\ldots (270.$$

or, with $\Delta l = 0$ and if x is the distance of the load from the abutment,

$$H = \frac{4x(l-x)}{l^2 \pi} \cdot G \qquad (270^a.$$

The H-influence line for a semicircular arch is therefore a parabola.

The equation of the reaction locus is derived from equ. (198^a):

$$\eta = \frac{Gr(\sin^2\beta - \sin^2\gamma)}{2H \sin\beta} \qquad (271.$$

For the semicircular arch, this reduces to

$$\eta = \frac{r\pi}{2} \qquad (271^a.$$

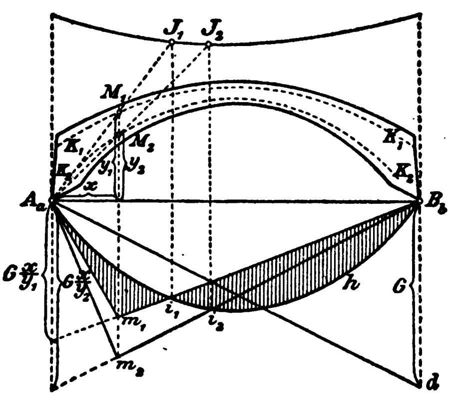
Hence the locus is, in this case, a horizontal line.

Flat semicircular arches may, with little error, be designed by the formulae for parabolic arches.

§ 18. Maximum Moments and Shears.

1. General Case. Graphic Method. There is nothing essential to be added to the general rules given in § 14, for the construction of the reaction locus and the determination of the critical loading, in applying them to the two-hinged arch. If the H-curve (or H-polygon, if the loads cannot be transferred to the arch except at distinct points) is plotted according to the values of H calculated for different positions of a moving load, or obtained graphically as described in § 17 : 1, then the reaction locus may be simply constructed by drawing the successive end-reactions, through the end hinges, as the resultants of the

Fig. 61.



corresponding values of H and V_1. With the aid of this locus, the critical loads may be determined according to § 14. In finding the maximum moments and shears, we may apply the principles given in § 7 : 3, 4. As before, we may either use the method of influence lines or construct the equilibrium polygon of the loading. As in the case of the three-hinged arch, it is advisable to take the moments about the core-points of the cross-sections since, by equ. (179), the extreme fiber stresses are directly determinable from these moments.

Fig. 61 shows the application of the method of influence lines. $K_1 K_1$ and $K_2 K_2$ are the two core-lines. Since, in general, $M = \mathbf{M} - H\,y = y\,\left(\frac{\mathbf{M}}{y} - H\right)$, the moment for a moving load will be represented by the difference between the ordinates of

the H-curve and those of the ordinary moment influence line for a simple beam, the latter figure being constructed with the distance y as the unit load. Since J_1 is the critical point (or load-position giving zero moment) for the upper core-point, and J_2 is the critical point for the lower core-point of the section $M_1 M_2$, the figures $a\ m_1\ i_1\ b$ and $a\ m_2\ i_2\ b$ will be the moment influence lines for the respective core-points; and if v_1 and v_2 are the distances of the upper and lower extreme fibers from the gravity axis, then, for a uniformly distributed live load of p per unit length, we find:

Maximum tension in the upper fiber $= \frac{v_1 y_2}{I} \cdot area\ \overline{i_2 h b} \left(\frac{pl}{2\ area\ \overline{adb}}\right)$

Maximum compression in the upper fiber $= \frac{v_1 y_2}{I} \cdot area\ \overline{a m_2 i_2} \left(\frac{pl}{2\ area\ \overline{adb}}\right)$

Maximum tension in the lower fiber $= \frac{v_2 y_1}{I} \cdot area\ \overline{a m_1 i_1} \left(\frac{p^l}{2\ area\ \overline{adb}}\right)$

Maximum compression in the lower fiber $= \frac{v_2 y_1}{I} \cdot area\ \overline{i_1 h b} \left(\frac{pl}{2\ area\ \overline{adb}}\right)$

The evaluation of the areas of the influence lines may be accomplished graphically, as already outlined in § 7 : 3 (a).

The influence line for shears is represented by Fig. 49 if the line $a\ b\ c$ is replaced by the H-curve.

The method of influence lines is particularly advantageous when the live load consists of a train of concentrations. In that case the position of loading for maximum stress may be found by the rules given on page 42.

It should also be noted that, usually, even continuous loading is transmitted to the main arch at distinct points. The influence lines are then no longer continuous curves but assume the form of polygonal figures with the vertices on the vertical lines through the points receiving the load.

The second graphic method, for determining the moments and shears by means of the equilibrium polygon for partial loading, is to be carried out exactly as described in § 7 : 3 (b).

2. Arch with Parabolic Axis and Variable Moment of Inertia. The horizontal thrust for a uniform load p covering the entire span is found by integrating equ. (265). If N denotes the denominator of that equation, we obtain

$$H_{tot} = \frac{\frac{1}{12} + \frac{1}{60}\mathbf{a} - \frac{1}{140}\mathbf{b} + \frac{1}{252}\mathbf{c}}{N} p \frac{l^2}{f} \ldots\ldots (272.$$

For a partial load, extending from one end to a point distant $\pm \xi'_1$ from the crown, if we put $\pm \frac{2\xi'_1}{l} = \pm \gamma_1$ (where the

+ sign applies if the load extends over the crown and the — sign if the load falls short of the crown), we find the horizontal thrust,

$$\mathbf{H}=\frac{\begin{array}{l}(2+3\gamma_1-\gamma_1^3)+\frac{1}{10}(4+5\gamma_1-\gamma_1^5)\,.\,\mathbf{a}\\ -\frac{1}{140}(6+7\gamma_1-\gamma_1^7)\,\mathbf{b}+\frac{1}{336}(8+9\gamma_1-\gamma_1^9)\,\mathbf{c}\end{array}}{48\,N}\,.\,p\frac{l^2}{f}\quad\ldots(273.$$

If x_k, y_k are the coordinates of the core-point of any cross-section, the limits of loading for maximum stress are determined by the conditions:

$$\frac{x_k}{y_k}=\frac{H}{V_1}\quad\text{and}\quad\frac{l-x_k}{y_k}=\frac{H}{V_2}.$$

Substituting the value for H from equ. (265), also $V_1=G\,\frac{1+\gamma}{2}$, we obtain,

$$\left.\begin{array}{l}\frac{1}{4}(1-\gamma_1)\Big[1+\frac{1}{6}(1+\gamma_1^2)\,\mathbf{a}-\frac{1}{15}(1+\gamma_1^2+\gamma_1^4)\,\mathbf{b}\\ \qquad+\frac{1}{28}(1+\gamma_1^2+\gamma_1^4+\gamma_1^6)\,\mathbf{c}\Big]\frac{l}{f}=\frac{x_k}{y_k}\,.\,N\end{array}\right\}\ldots\ldots(274.$$

The value of γ_1 given by solving this equation is to be used in calculating $\mathbf{H}$ by equ. (273); we may then find the greatest negative moment by the relation

$$M_{\min}=\mathbf{V}\,.\,x_k-\mathbf{H}\,.\,y_k=\frac{1}{8}(1+\gamma_1)^2plx_k-\mathbf{H}\,.\,y_k\ldots\ldots\ldots\ldots(275.$$

This expression holds good for all sections from the end to a point whose abscissa is given by

$$l-x'_k=\frac{l\left(1+\frac{1}{3}\mathbf{a}-\frac{1}{5}\mathbf{b}+\frac{1}{7}\mathbf{c}\right)}{2\,Nf}\,.\,y'_k\quad\ldots\ldots(276.$$

For the middle portion from x'_k to $l-x'_k$, for which an interrupted load extending on each side to the end of the span must be taken, there is to be added to the value of M given by equ. (275) the value given by the same formula for the symmetrically located point of the arch.

The general formulae may be used for calculating the shears.

3. Arch with Flat Parabolic Axis and Parallel Flanges. This is merely a special case of the preceding one; if the moment of inertia is constant throughout, the corresponding formulae may be obtained by simply substituting $\epsilon=\chi=0$ and $\nu=1$ in equs. (272) to (276). If the moment of inertia increases toward the abutments with the secant of the slope, i. e., if $I_x=$

$I_0 \sec \phi$, we obtain the following expressions directly by integrating equ. (261^a):

For a full-span load [*cf.* (82)],

$$\mathbf{H}_{tot}=\frac{1}{8}\frac{pl^2}{f'}. \qquad (277.$$

For a load covering a length λ_1 from one end [*cf.* equ. (83)],

$$\mathbf{H}=\frac{5}{16}\left(\frac{\lambda_1}{l}\right)^2\left[1-\left(\frac{\lambda_1}{l}\right)^2+\frac{2}{5}\left(\frac{\lambda_1}{l}\right)^3\right]\frac{pl^2}{f'} \qquad (278.$$

For a crown-load of length a,

$$\mathbf{H}=\frac{1}{128}\left(25-10\frac{a^2}{l^2}+\frac{a^4}{l^4}\right)\frac{l}{f'}\cdot pa \qquad (279.$$

For the greatest negative moment, λ_1 is determined by

$$(l-\lambda_1)(l^2+l\lambda_1-\lambda_1^2)=\frac{8}{5}\frac{x_k}{y_k}f'\,l^2 \qquad (280.$$

and the value of the moment is given by

$$M_{min}=\mathbf{V}\,.\,x_k-\mathbf{H}\,.\,y_k=\frac{p\lambda_1^2}{2\,l}\,.\,x_k-\mathbf{H}\,.\,y_k \qquad (281.$$

For the central portion included between the abscissae x'_k and $l-x'_k$, determined by the equation

$$l-x'_k=\frac{5}{8}\frac{l}{f'}\cdot y'_k \qquad (282.$$

we must add to the above values of M the values given by the same formula for the symmetrically located points of the arch.

For a full-span load, the moment about any point of the parabolic axis is given by

$$M_{tot}=\frac{px(l-x)}{2}\cdot\frac{f'-f}{f'} \qquad (283.$$

or, if the moments are taken about the core-points,

$$M_{tot}=\frac{px_k(l-x_k)}{2}-\frac{pl^2}{8f'}\cdot y_k \qquad (283^a.$$

From the above values, the maximum positive moments may readily be deduced.

If f', defined by equ. (262) or approximately by

$$f'=f\left(1+\frac{15}{8}\frac{I_0}{A_0f^2}\right),$$

is put equal to f, which is equivalent to neglecting the effect of the axial force upon the deformations, then we should have $\mathbf{H}_{tot}=\frac{1}{8}\frac{pl^2}{f}=\mathbf{H}'_{tot}$ and the moment about any point of the parabolic axis $=0$; hence,

under this assumption, the line of pressures coincides with the axis of the arch. The error thus introduced into the determination of the stresses, especially near the crown, may sometimes become considerable. For example, if the cross-section at the crown consists of two flanges having a combined area A_0, then, with the above approximation, the intensity of stress in each flange will be $\sigma' = -\frac{H'}{A_0}$. But if we substitute the value $f' = f\left(1 + \frac{15}{32} \cdot \frac{h^2}{f^2}\right)$, we find:

The stress in the upper flange,

$$\sigma_u = \sigma' \frac{f}{f'}\left(1 + \frac{2\,(f'-f)}{h}\right) = \sigma' \frac{1 + \frac{15}{16}\frac{h}{f}}{1 + \frac{15}{32}\frac{h^2}{f^2}};$$

The stress in the lower flange,

$$\sigma_l = \sigma' \frac{f}{f'}\left(1 - \frac{2\,(f'-f)}{h}\right) = \sigma' \frac{1 - \frac{15}{16}\frac{h}{f}}{1 + \frac{15}{32}\frac{h^2}{f^2}}.$$

The former attains its maximum value with $\frac{h}{f} = 0.742$, wher $\sigma_u = 1.35\,\sigma'$ and $\sigma_l = 0.242\,\sigma'$. With $\frac{h}{f} = \frac{1}{3}$, we find for the upper flange, $\sigma_u = 1.24\,\sigma'$, and for the lower flange, $\sigma_l = 0.65\,\sigma'$.

The critical loading for shears is determined by

$$\lambda^2\,(l^2 + l\,\lambda_2 - \lambda_2^2) = \frac{8}{5} f'\, l^2 \cot\phi \,.\,.\,.\,.\,.\,.\,.\,.\,.\,. (284.$$

and, if $H_{x-\lambda_2}$ denotes the horizontal thrust calculated by equ. (278) for a load extending from x to λ_2, the maximum + shear will be:

$$S_{max} = \frac{p}{2l}\,[\,(l-x)^2 - (l-\lambda_2)^2]\cos\phi - H_{x-\lambda_2} \,.\, \sin\phi \,.\,. (285.$$

and, similarly,

$$S_{min} = \frac{1}{2}\,p\,(l-2x)\cos\phi - \frac{p\,l^2}{8\,f'} \,.\, \sin\phi - S_{max} \,.\,.\,.\,.\,.\,. (286.$$

4. Segmental Arch of Uniform Cross-Section. The horizontal thrust for a continuous, uniformly distributed load p may be obtained by integrating equ. (269) between the appropriate limits, after substituting $G = p\,.\,dx = -p\,r\cos\gamma\,.\,d\gamma$. If the load extends from one end to a point departing from the crown by the central angle γ, the limits to be used are $\gamma = \beta$ and γ, so that we obtain:

$$H = \frac{\sin\beta.\mathbf{A} - \sin\gamma.\mathbf{B} + \frac{1}{2}\cos\beta.(\beta-\gamma) - \frac{1}{2}\delta.\cos\beta.\mathbf{C}}{2\,(\beta - 3\sin\beta\cos\beta + 2\,(1+\delta)\,\beta\cos^2\beta)} \,.\, p\,r \ldots (287.$$

Here,

$$\mathbf{A} = \frac{2}{3}\sin^2\beta - \frac{1}{2}\cos^2\beta - \beta\sin\beta\cos\beta,$$

$$\mathbf{B} = \sin^2\beta - \frac{1}{3}\sin^2\gamma - \cos\beta\left(2\cos\beta + 2\beta\sin\beta - \gamma\sin\gamma - \frac{3}{2}\cos\gamma\right),$$

$$\mathbf{C} = \sin\beta\,(2\beta\sin\beta - 4\beta\sin\gamma - \cos\beta) + \sin\gamma\,(2\gamma\sin\gamma + \cos\gamma) + \beta - \gamma.$$

If the entire span is loaded, $\gamma = -\beta$, and we have,

$$H_{tot} = \frac{\sin^2\beta - 3\,(1-2\sin^2\beta)\,(\sin\beta - \beta\cos\beta) - 3\delta\cos\beta(\beta + 2\beta\sin^2\beta - \sin\beta\cos\beta)}{6(\beta - 3\sin\beta\cos\beta + 2\,(1+\delta)\,\beta\cos^2\beta)}\, p\,r \ldots\ldots (288.$$

and, if the arch is a complete semicircle,

$$H_{tot} = \frac{4}{3\pi}\,.\,p\,r \ldots\ldots\ldots\ldots (289.$$

Fig. 62.

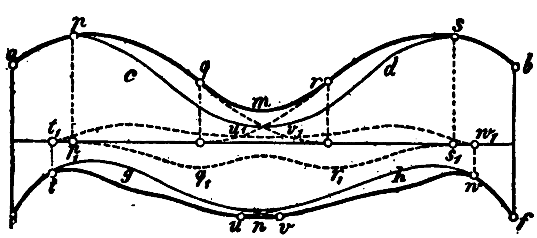
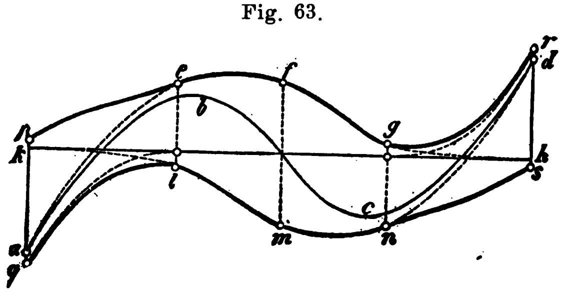


Fig. 63.



With the aid of the reaction locus given by equ. (271) we may determine the critical loadings to be used in finding the maximum moments and shears by equs. (281) and (285) respectively. These are plotted in Figs. (62) and (63) for a

segmental arch of uniform core-depth. The same notation is here used as for the three-hinged arch in Figs. (50) and (51), so that we may refer to the explanation there given (p. 113).

5. Temperature Stresses. The horizontal thrust, H_t, produced by a uniform change of temperature, t, is given by equs. (250) to (269), if we substitute $G=0$, $\mathbf{M}=0$, and $\Delta l=0$. For structural steel we may take $E\omega=196$ lbs. / sq. in. / °F. (= 248 tonnes / sq. m. / °C.) and, for the variation of temperature, assume $t=\pm 55$°F. Equ. (250) then becomes:

$$H_t = \pm 10{,}780 \frac{I_0 \,.\, b \,.\, \cos\beta}{a_0 \sum\limits_0^l y_m \,.\, v_m + \dfrac{I_0 b \cos\beta}{A_0}} \;\ldots\ldots (290.$$

or, for flat parabolic arches of uniform cross-section, equ. (261) gives approximately,

$$H_t = \pm 10{,}780 \frac{15}{8 f^2} \cdot \frac{I_0}{1+\dfrac{15}{8}\dfrac{I_0}{A_0 f^2}} \;\ldots\ldots (290^a.$$

In the crescent rib of parabolic form, equ. (266) shows that the value of H is reduced to about half the above value.

The moments due to temperature change, referred to the core-points, are given by

$$M_t = -H_t \,.\, y_k \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (291.$$

These moments are thus proportional to the distances of the core-points from the closing chord, and hence they attain their greatest value at the crown.

The shear due to temperature variation is, by equ. (192),

$$S_t = -H_t \,.\, \sin\phi \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (292.$$

In an arched rib consisting of two equal flanges, if A is the total cross-section and h the effective depth, the extreme fiber stresses due to temperature variation are, by (179),

$$\sigma = \pm \frac{H_t\left(y \mp \dfrac{h}{2}\right)\dfrac{h}{2}}{I} = \pm \frac{2 H_t \left(y \mp \dfrac{h}{2}\right)}{A \,.\, h}$$

and, introducing the value of H_t from equ. (290^a), we obtain, for the parabolic arch with parallel flanges,

$$\sigma = \pm E\omega t \cdot \frac{15}{32} \cdot \frac{h\,(2y \mp h)}{f^2\left(1+\dfrac{15}{32}\dfrac{h^2}{f^2}\right)}.$$

Accordingly, with $E\omega t = 10{,}780$ lbs. per sq. in., the extreme fiber stresses at the crown will be:

$$\sigma = \pm 10{,}000\,\frac{h}{f}\,\frac{1 \mp \frac{1}{2}\frac{h}{f}}{1 + \frac{15}{32}\frac{h^2}{f^2}}.$$

The stress in the upper flange is a maximum with $\frac{h}{f} = 0.742$; the following values then obtain:

$$\begin{cases} \sigma_u = \pm\, 0.371 \times 10{,}000 = \pm\, 3{,}710 \text{ lbs. / sq. in.} \\ \sigma_l = \mp\, 0.808 \times 10{,}000 = \mp\, 8{,}080 \text{ lbs. / sq. in.} \end{cases}$$

With a ratio of $\frac{h}{f} = \frac{1}{3}$,

$$\begin{cases} \sigma_u = \pm\, 2{,}640 \text{ lbs. / sq. in.} \\ \sigma_l = \mp\, 3{,}700 \text{ lbs. / sq. in.} \end{cases}$$

In the case of the parabolic crescent arch, the temperature stresses are constant throughout each flange, and their approximate value is

$$\sigma = \pm\, 5{,}320\,\frac{h}{f}\,\frac{1 \mp \frac{1}{2}\frac{h}{f}}{1 + \frac{1}{4}\frac{h^2}{f^2}}$$

Hence, with the same ratio of crown-depth to rise, the temperature stresses in the crescent arch are less than in the arch with parallel flanges.

In an arch having its ends connected with a tie-rod of the same material as the rib itself, no stresses are produced by uniform changes of temperature.

§ 19. Deformations.

To determine the elastic deflections of any point (x_1, y_1) of the axis of the arch, equations (185) and (186) may be employed. Let us first apply equ. (186) to the end-points; simplifying by writing $P' = P$ and $J = I$, and assuming a symmetrical arch-form, if $b =$ the total length of axis, we find for the rotation at the end A,

$$\Delta \phi_0 = -\frac{1}{l}\int_0^b \frac{M}{EI}(l-x)\,.ds - \frac{1}{l}\int_0^b \left(\frac{P}{EA} - \omega t\right)\left[(l-x)\frac{ds}{r} - dy\right]$$

Again substituting $I' = I \cos\phi = I\frac{dx}{ds}$, and writing, for abbreviation,

$$\left[M\frac{I_0}{I'} + \left(\frac{P}{A} - E\omega t\right)\frac{I_0}{r\cos\phi}\right] = z, \quad \ldots\ldots. (293.$$

the above equation assumes the form:

$$EI_0\,.\,\Delta\phi_0 = -\frac{1}{l}\int_0^l z(l-x)\,dx - \frac{I_0}{l}\int_{x=0}^{x=l}\left(\frac{P}{A} - E\,\omega\,t\right)dy \,..\, (294.$$

Substituting this relation in equations (185) and (186), we obtain the following expressions for the variations of the co-ordinates:

$$-EI_0\,.\,\Delta y = \frac{x_1}{l}\int_0^l z(l-x)\,dx - \int_0^{x_1} z(x_1 - x)\,dx + C_1 \ldots (295.$$

$$EI_0\,.\,\Delta x = \frac{y_1}{l}\int_0^l z(l-x)\,dx - \int_0^{x_1} z(y_1 - y)\,dx + C_2 \ldots. (296.$$

where

$$\left.\begin{aligned} C_1 &= -\frac{x_1}{l} I_0 \int_{x=0}^{x=l}\left(\frac{P}{A} - E\omega t\right).dy + I_0\int_0^{y_1}\left(\frac{P}{A} - E\omega t\right).dy \\ C_2 &= +\frac{y_1}{l} I_0 \int_{x=0}^{x=l}\left(\frac{P}{A} - E\omega t\right).dy - I_0\int_0^{x_1}\left(\frac{P}{A} - E\omega t\right).dx \end{aligned}\right\} .. (297.$$

represent the contribution of the axial compression to the total deformation. If we adopt the usually admissible approximation of putting $\frac{P}{A} = \frac{H}{A_0}$ = constant, where A_0 is a mean area of section, and if t is supposed uniform throughout the span, then the second definite integral of equ. (294) vanishes and we have

$$\left.\begin{aligned} C_1 &= I_0\left(\frac{H}{A_0} - E\,\omega\,t\right)y_1 \\ C_2 &= -I_0\left(\frac{H}{A} - E\,\omega\,t\right)x_1 \end{aligned}\right\} \quad \text{(297}^{a}\text{.}$$

The definite integral containing the quantity z may readily be given a static interpretation: If the quantities z are considered as forces acting on the horizontal projection of the arch-axis, then the second member of equ. (294) represents the end-reaction Z_a for these forces. Let M_{zv} and M_{zh} denote the moments of the forces Z_a and $\sum\limits_0^{x_1} z$ about the arch point (x_1, y_1), when these forces are applied vertically and horizontally, respectively, at their respective points. Then we have,

$$E\,I_0\,\Delta\,\phi_0 = Z_a \quad \text{(294}^{a}\text{.}$$

$$-E\,I_0\,\Delta\,y = M_{zv} + C_1 \quad \text{(295}^{a}\text{.}$$

$$E\,I_0\,\Delta\,x = M_{zh} + C_2 \quad \text{(296}^{a}\text{.}$$

If the radius of curvature of the arch axis is very large relative to the sectional dimensions, we may write $z = \frac{I_0}{I'} \cdot M$; hence, neglecting the term C_1 which depends upon the axial compression, we may obtain the vertical deflections $(-\Delta\,y)$ of the arch as the ordinates of a funicular polygon, constructed with the pole distance $E\,I_0$, for a loading consisting of the area of the curve of reduced moments, $M\,\frac{I_0}{I'}$. With the above assumptions, the deflections may be determined by rules which are practically the same as those for a simple beam.

For finding the deflections graphically, a treatment suggests itself similar to that applied to suspension bridges in § 8. If we write, with the usual notation, $M = \mathbf{M} - Hy$, there results

$$z = H\left[\left(\frac{\mathbf{M}}{H} - y\right)\frac{I_0}{I'} + c\right], \quad \text{(298.}$$

where

$$c = \left(1 - \frac{E A_0 \omega t}{H}\right) \frac{I_0}{A_0 r \cos\phi} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (299.$$

The moments, M_{zv} and M_{zh}, may then be represented by the differences between the ordinates of two funicular polygons: one constructed for the supposed loads $\frac{\mathbf{M}}{H} \cdot \frac{I_0}{I'} + c$, and the other for the supposed loads $y \cdot \frac{I_0}{I'}$. The former polygon is obtained by reducing the ordinates of the simple moment curve, constructed with pole distance H, in the ratio $\frac{I_0}{I'}$ and adding thereto the small and nearly constant quantities c; while the funicular polygon for the loads $y \cdot \frac{I_0}{I'}$ is identical with the H-influence line described in § 17:1 and constructed as shown in Fig. 54.

Little need here be added to make clear the graphic determination of the deflections of a crescent-shaped arched rib as executed in Plate I, Figs. 1 to 1^h. In Fig. 1, the ordinates of the arch-axis are multiplied by the ratio $\frac{I_0}{I'}$; and from the resulting values, with the aid of the force polygon Fig. 1^b, there is constructed the funicular polygon I, i. e., the H-curve for a moving concentration G. The value of this unit load G is determined in the familiar manner by the funicular polygon II; in the example selected, with $l = 160$ meters, $I_0 = 2.80 \text{ m.}^4$, $A_0 = 0.112$ sq. m., $a = 16$ m., $p = 180$ m., the necessary correction $nn_1 = \frac{b \cos\beta \cdot I_0}{A_0 \cdot a_0 \cdot p}$ $= 1.13$ m. We proceed to find the vertical deflections of the points of the arch produced by a load $G = 1$ applied at the point C (x_1, y_1). The load values z are derived from the simple moment diagram (Fig. 1^d) constructed from the force polygon Fig. 1^e with the pole distance H. The ordinates of this moment diagram are multiplied by $\frac{I_0}{I'}$ and should be increased by the quantities $c = \frac{I_0}{A_0 r_0} = 0.33$ m.; but these corrections are vanishingly small on the scale of the drawing. From the resulting ordinates there are constructed, with the aid of the force polygon Fig. 1^f, the funicular polygon III and, for horizontal action of the forces, the funicular polygon IV. With the provisional neglect of the effects of the axial compression, the *vertical* intercepts between the funicular polygons I and III and the *horizontal* intercepts between the funicular polygons II and IV, or rather between the corresponding funicular curves, are proportional respectively to the vertical and horizontal deflections of the points of the arch under the given loading; and the scale for these deflections is $\frac{E I_0}{H \cdot a \cdot p}$ times the scale of lengths. (In the example on Plate I, the scale of lengths is 0.5 mm. $= 1$ m., $G = 1$ t., $H = 0.454$ t., $E = 20{,}000{,}000$ t. per sq. m.; hence the scale for the deflections is 1 m. $= \frac{20{,}000{,}000 \times 2.8}{0.454 \times 16 \times 180} \times 0.5$

mm., or 1 mm. in the structure $= 21.4$ mm. on the drawing: Scale I.) The corrections $\frac{C_1}{E I_0}$ and $-\frac{C_2}{E I_0}$, representing the coordinate components of the deflections due to the axial forces, may be assumed proportional to the coordinates x and y; the corresponding scale is $\frac{E A_0}{H}$ times the scale of lengths. Thus, in the example, the scale is 1 m. $= \frac{20{,}000{,}000 \times 0.112}{0.454}$ $\times 0.5$ mm. or, with $G = 1$ t., 1 mm. in the structure $= 2{,}467$ mm. on the drawing. It thus appears that the effect of the axial compression, compared with the deflections first investigated, is an absolutely negligible quantity.

The intercepts between the two funicular polygons I and III serve not only to determine the deflections at all points produced by a unit load (G) at C but also, by Maxwell's Theorem of Reciprocal Deflections, the intercept at any point M will give the deflection at C produced by a unit load at M. Consequently the above intercepts constitute the *influence line* for the *vertical* deflections at the point C; and, with this, the total deflection at C produced by any given loading may be obtained by the usual rules.

Similarly, the influence line for the *horizontal* deflections of the point C may be represented by the intercepts between the funicular polygons I and V (Fig. 1[h]), if we apply the principle that the horizontal deflection of C produced by a unit vertical load at any point M is equal to the vertical deflection of M produced by a unit horizontal load at C. The latter deflection, however, may be constructed, as before, by finding the simple moments of the horizontal load and treating the resulting values of $\mathbf{M}_w \cdot \frac{I_0}{I'}$ as vertically applied forces. The moments $\mathbf{M}_w$ for $W = 1$, defined by $\mathbf{M}_w = y - x\,\frac{y_1}{l}\Big]_{x=0}^{x=x_1}$ and $\mathbf{M}_w = (l - x)\,\frac{y_1}{l}\Big]_{x=x_1}^{x=l}$, are easily constructed; and from the ordinates of this diagram, multiplied by $\frac{I_0}{I'}$, there are derived the force polygon (Fig. 1[g]), whose pole distance is $H_{y_1}\,\frac{p}{G}$, and the funicular polygon V. The differences between the ordinates of this polygon and those of the H-curve (Funicular Polygon I) determine the horizontal deflection of the point C for any position of the moving load; the corresponding scale is $\frac{E I_0}{H_{y_1} a\, p}$ times the scale of lengths. (In the example, $H_{y_1} = 0.405$, hence the scale for the horizontal deflections is 1 mm. in the structure $= 23.9$ mm. on the drawing. Scale II.)

The influence lines for the quantities C_1 and C_2, i. e., the deflections due to axial compression which should be added to the above deflection values, may be represented, according to equ. (297[a]), by the H-curve; and the scales for measuring this curve, to obtain the vertical and horizontal deflections of the point C, are $\frac{E A_0}{y_1}$ and $\frac{E A_0}{x_1}$ times the force-unit, respectively. Thus, in the example, the scale for vertical deflections is 1 mm. in the structure $= \frac{20{,}000 \times 0.112}{27.2} \times 37.5 = 3082$ mm. on the drawing (Scale III); and the scale for horizontal deflections

is 1 mm, in the structure $= \dfrac{20{,}000 \times 0.112}{32} \times 37.5 = 2625$ mm. on the drawing (Scale IV).

If the arch has a uniform moment of inertia, there may be deduced from equ. (295ᵃ) a direct expression for the deflection producible at the point (x, y) by a concentration G placed at a distance ξ from the end of the span. If, as an approximation, we write $\dfrac{1}{r\cos\phi} = \dfrac{1}{r_0}$, equ. (295ᵃ) yields the following: (*cf.* equ. 131ᵃ.)

$$-EI_0 . \Delta y = G\mathrm{m}_x - H . m_x + Hc\left(\frac{x(l-x)}{2} + r_0 y\right) \ldots (300.$$

Here, as in equs. (130),

$$\left.\begin{aligned} \mathrm{m}_x &= (2l\xi - \xi^2 - x^2)\,\frac{x(l-\xi)}{6l}, \text{ for } x < \xi \\ \mathrm{m}_x &= (2lx - x^2 - \xi^2)\,\frac{\xi(l-x)}{6l}, \text{ for } x > \xi \end{aligned}\right\} \ldots (301.$$

H is the horizontal thrust to be determined as in § 17; and, as in (132),

$$m_x = \frac{l-x}{l}\int_0^x xy\,dx + \frac{x}{l}\int_x^l (l-x)\,y\,dx \ldots\ldots (302.$$

In general, with variable moment of inertia, m_x is proportional to the ordinates of the H-curve.

With a parabolic arch-axis,

$$m_x = \frac{fx}{3l^2}(x^3 - 2lx^2 + l^3) \ldots\ldots\ldots\ldots\ldots (303.$$

and

$$-EI_0\Delta y = G\mathrm{m}_x - Hm_x + Hcx\,(l-x) \ldots\ldots (304.$$

The vertical deflection of the arch-point (x, y) due to temperature variation may be obtained from equ. (300) by putting $G = 0$ and $H =$ the value of H_t given by equ. (290). Hence,

$$EI_0\Delta y = H_t . m_x + 2\left(E\omega t - \frac{H_t}{A_0}\right)I_0 y \ldots\ldots\ldots (305.$$

In the example executed in Plate I, assuming $E\omega = 248$ tonnes / sq. m. / °C., we have $H_t = 248 . t \dfrac{2.8 \times 128}{16 \times 180 \times 75} = 0.411 . t$ tonnes. If η is the ordinate of the H-curve (Funicular Polygon I) measured to the scale of lengths, then $m_x = \eta . p . a = 180 \times 16 \times \eta$; consequently $EI_0\Delta y = (1184.6\,\eta + 1368.3\ y)t$ or, finally, Δy (in mm.) $= (0.02116\,\eta + 0.02444\,y)\,t$, where η and y, measured to the scale of lengths, must be

given in meters. For $t = \pm 30°C.$, the vertical deflection at the crown amounts to

$$\Delta y_c = \pm (0.02116 \times 51.75 + 0.02444 \times 42.5)\ 30 = \pm 64 \text{ mm.}$$

Similarly, in the case of a displacement of the abutments increasing the length of span by Δl, the vertical deflection of the arch-point (x, y), is given by

$$\Delta y = -\frac{\Delta l}{N}\left(m_x - 2\frac{I_0}{A_0} \cdot y\right) \ldots\ldots\ldots\ldots\ldots\ldots (306.$$

where

$$N = \int_0^b y^2 \cdot \frac{I_0}{I} \cdot ds + \frac{I_0 b \cos\beta}{A_0},$$

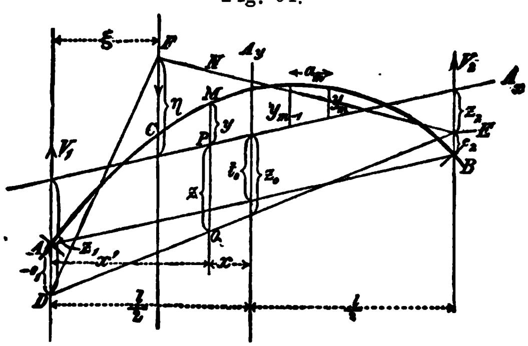
i. e., with reference to the graphic representation, $N = \overline{n_0 n_1} \cdot a \cdot p$.

3. ARCHED RIB WITHOUT HINGES.

§ 20. Determination of the Reactions.

1. General Case. Concentrated Load. We assume an arch so rigidly supported that its end sections cannot undergo rotation; hence the ends are to be considered as fixed. In general, the ends may lie at different elevations; but the form of the axis must be such that the vertical center line bisects all chords parallel to the closing chord.

Fig. 64.



The fundamental equations are obtained from equs. (185) and (186) by extending their limits of integration over the entire arch and introducing the condition $\Delta \phi_1 = 0$ which must be satisfied on account of the fixedness of the ends. The horizontal and vertical coordinates (x, y) of the points of the arch are to be measured from a pair of axes so located that the Y-axis coincides with the vertical center-line of the arch while the X-axis is parallel to the closing-chord at a vertical distance t_0 above that chord. With the admissible simplifications $J = I$, $P' = P$, and provisional disregard of the effect of temperature, also assuming an unstressed initial condition of the rib, the general equations become:

$$\left.\begin{aligned} \Delta \phi_1 = 0 &= \int_0^b \frac{M}{EI} ds + \int_0^b \frac{P}{EAr} \cdot ds \\ -\Delta x_1 = 0 &= \int_0^b \frac{M}{EI} y\, ds + \int_0^b \frac{P}{EA}\left(\frac{y}{r} ds - dx\right) \\ \Delta y_1 = 0 &= \int_0^b \frac{M}{EI} x\, ds + \int_0^b \frac{P}{EA}\left(\frac{x}{r} ds + dy\right) \end{aligned}\right\}$$

Observing the smallness of the terms containing P in comparison with those in which M occurs, it appears fully permissible to put $P = H$ and to consider A = constant = an average cross-section A_0. We then have, either exactly or very nearly,

$$\left.\begin{array}{l} \int\limits_0^b \frac{P.ds}{EA\,r} = \frac{H.b}{EA_0 r_0} \\ \int\limits_0^b \frac{P}{EA}\left(\frac{y}{r}\,ds - dx\right) = -\frac{Hl}{EA_0} \\ \int\limits_0^b \frac{P}{EA}\left(\frac{x}{r}\,ds + dy\right) = 0 \end{array}\right\}$$

so that, upon replacing I by $I' = I\cos\phi = I\,\frac{dx}{ds}$, (where x is not measured parallel to the axis A_x but horizontally), we obtain:

$$\left.\begin{array}{l} \int\limits_{-\frac{l}{2}}^{+\frac{l}{2}} \frac{M}{I'}.dx + \frac{Hb}{A_0 r_0} = 0 \\ \int\limits_{-\frac{l}{2}}^{+\frac{l}{2}} \frac{M}{I'}.y.dx - \frac{Hl}{A_0} = 0 \\ \int\limits_{-\frac{l}{2}}^{+\frac{l}{2}} \frac{M}{I'}.x\,dx = 0 \end{array}\right\} \quad \ldots\ldots\ldots\ldots (307.$$

As the hingeless arch is triply indeterminate with respect to the external forces, the above three equations of condition will suffice for the evaluation of the unknown reaction elements.

As proved in § 13, the moment M is given by the vertical intercept between the equilibrium polygon of the loading and the axis of the arch, multiplied by H. Consequently, if DFE (Fig. 64) is the funicular polygon for a concentration applied at any point C, constructed with H as the pole distance, we have:

$$M = H.\overline{MN} = H(\overline{QN} - \overline{PM} - \overline{PQ}) = \mathbf{M} - H(y + z)$$

where **M**, as before, is the moment which would be produced in a freely supported beam of span l. With $z=z_0+\frac{e_2-e_1}{l}\,.\,x$, there results:

$$M=\mathbf{M}-Hy-Hz_0-H\frac{e_2-e_1}{l}\,.\,x$$

As unknowns whose values are necessary and sufficient to fully determine the equilibrium polygon, let us choose*

$$H,\ X_1=H\cdot\frac{e_2-e_1}{l},\ \text{and}\ X_2=Hz_0 \qquad (308.$$

The expression for the bending moment then becomes:

$$M=\mathbf{M}-Hy-X_1\,.\,x-X_2 \qquad (309.$$

Upon substituting this value in equs. (307) and solving them for the three unknowns, remembering that, on account of the assumed (oblique) symmetry of the arch about the Y-axis,

$$\int_{-\frac{l}{2}}^{+\frac{l}{2}}\frac{x\,dx}{I'}=0,\quad\text{also}\quad\int_{-\frac{l}{2}}^{+\frac{l}{2}}\frac{xy\,dx}{I'}=0,$$

and if, furthermore, we choose the X-axis that

$$\int_{-\frac{l}{2}}^{+\frac{l}{2}}\frac{y\,dx}{I'}=0, \qquad (310.$$

we obtain the following expressions:

$$H=\frac{\displaystyle\int_{-\frac{l}{2}}^{+\frac{l}{2}}\frac{\mathbf{M}}{I'}\,y\,dx}{\displaystyle\int_{-\frac{l}{2}}^{+\frac{l}{2}}\frac{y^2dx}{I'}+\frac{l}{A_0}} \qquad (311.$$

* We here follow, in the principal features, the method of design of *F. B. Müller-Breslau.* (See Zeitschr. d. Arch. u. Ing. Ver. zu Hannover, 1884.)

$$X_1 = \frac{\int_{-\frac{l}{2}}^{+\frac{l}{2}} \frac{M}{I'} x\,dx}{\int_{-\frac{l}{2}}^{+\frac{l}{2}} \frac{x^2\,dx}{I'}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (312.$$

$$X_2 = \frac{\int_{-\frac{l}{2}}^{+\frac{l}{2}} \frac{M\,dx}{I'} + \frac{Hb}{A_0 r_0}}{\int_{-\frac{l}{2}}^{+\frac{l}{2}} \frac{dx}{I'}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (313.$$

The definite integrals appearing in the above expressions may be replaced by summations in a manner similar to that deduced for the integrals occurring in the expressions for H in § 5 and § 17. For this purpose we divide the arch into horizontal panel-lengths a_{m-1}, a_m, a_{m+1} ......, within which the moments of inertia of the cross-sections may be assumed constant and equal to I_{m-1}, I_m, I_{m+1} .... We introduce an arbitrary constant panel-length a_0, also a mean value for the moment of inertia, I_0, and adopt the following notation:

$$\begin{cases} v_m = \dfrac{a_m}{6a_0} \cdot \dfrac{I_0}{I'_m} (2y_m + y_{m-1}) + \dfrac{a_{m+1}}{6a_0} \dfrac{I_0}{I'_{m+1}} (2y_m + y_{m+1}) \;.. (314. \\[2ex] v'_m = \dfrac{a_m}{6a_0} \cdot \dfrac{I_0}{I'_m} (2x_m + x_{m-1}) + \dfrac{a_{m+1}}{6a_0} \cdot \dfrac{I_0}{I'_{m+1}} (2x_m + x_{m+1}) \;.. (315. \\[2ex] v''_m = \dfrac{1}{2} \dfrac{a_m}{a_0} \cdot \dfrac{I_0}{I'_m} + \dfrac{1}{2} \dfrac{a_{m+1}}{a_0} \dfrac{I_0}{I'_{m+1}} \ldots\ldots\ldots\ldots\ldots\ldots (316. \end{cases}$$

or, with sufficient approximation,

$$v'_m = x_m \cdot v''_m \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (317.$$

Substituting the above values in equs. (311) to (313), [*cf.* equs. (253) and (254)], we obtain:

$$\begin{cases} H = \dfrac{\Sigma M_m v_m}{\Sigma y_m v_m + \dfrac{I_0 l}{A_0 \cdot a_0}} \ldots\ldots\ldots\ldots\ldots\ldots (318. \\[3ex] X_1 = \dfrac{\Sigma M_m v'_m}{\Sigma x_m v'_m} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (319. \\[3ex] X_2 = \dfrac{\Sigma M_m v''_m + H \dfrac{I_0 b}{A_0 \cdot a_0 r_0}}{\Sigma v''_m} \ldots\ldots\ldots\ldots\ldots\ldots (320. \end{cases}$$

The summations appearing in these expressions should extend over the entire arch, hence from $-\frac{l}{2}$ to $+\frac{l}{2}$.

The position of the axis of abscissae, from which the ordinates y are measured, is fixed by equ. (310) or, if y' denotes the arch ordinates measured from the closing chord, the axis is determined by

$$t_0 = \frac{\int \frac{I_0}{I'} y' \, dx}{\int \frac{I_0}{I'} \cdot dx} = \frac{\sum_0^l y'_m v''_m}{\sum_0^l v''_m} \quad \ldots\ldots\ldots\ldots \quad (321.$$

For a constant value of I′, the X-axis becomes the rectifying line for the arch-curve; so that t_0 represents the altitude of a parallelogram erected upon the arch-chord with an area equal to that included between the chord and the arch-curve.

We proceed to determine the influence lines for the quantities H, X_1 and X_2 for a moving concentration. The graphic method may here be applied, since the summations appearing in the above expressions are readily represented by the ordinates of funicular polygons. Thus, if the load consists of a unit concentration applied at a distance ξ from one end, we have, as shown in § 5, p. 36, the following relation:

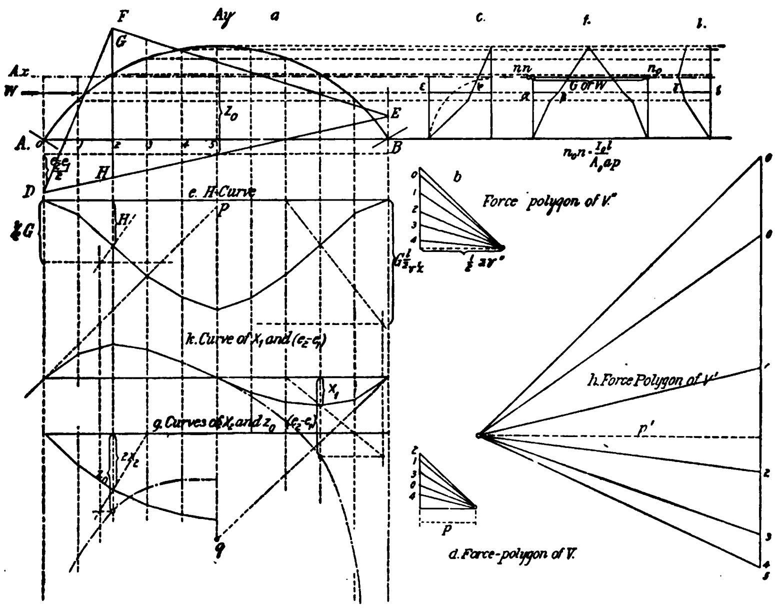
$$\Sigma \, \mathbf{M}_m v_m = \frac{l-\xi}{l} \sum_0^{\xi} x'_m v_m + \frac{\xi}{l} \sum_{\xi}^{l} (l - x'_m) v_m = \mathbf{M}_v$$

where $\mathbf{M}_v$ is the moment producible at the section ξ of a beam freely supported at A and B by a loading consisting of the vertical "forces" v_m. In similar manner we obtain the quantities $\sum_0^l \mathbf{M}_m v'_m$ and $\sum_0^l \mathbf{M}_m v''_m$ as the moments $\mathbf{M}_{v'}$ and $\mathbf{M}_{v''}$ of the "forces" v'_m and v''_m, respectively. Since the values of v_m, v'_m and v''_m may be calculated directly by equs. (314) to (316), there is no difficulty in constructing the above funicular polygons and, hence, the influence lines for H, X_1 and X_2.

In Figs. (65^a) to (65^k), this construction is carried out. We first have to find the rectifying axis A_x, for which we use the force polygon for the quantities v'' (Fig. 65^b) and the resulting funicular polygon constructed upon the arch-rise (Fig. 65^c). The calculated quantities v give the force polygon (Fig. 65^d) and, corresponding to the vertical and horizontal directions of application of these forces, we obtain the funicular polygons (Fig. 65^e) and (Fig. 65^f). The intercept of the latter polygon on the X-axis, augmented by the quantity $\frac{I_0 \, l}{A_0 \, a_0 \, p}$, gives the

magnitude of the unit force G; while the ordinates of the former polygon give the thrusts H. Constructing, next, the funicular polygon for the loads v'' considered as acting vertically on the straight beam $A\,B$, we obtain (in Fig. 65ᵍ) the influence line for the quantity X_2. Similarly the funicular polygon of the forces v' (Fig. 65ᵏ), constructed with the aid of the corresponding force polygon (Fig. 65ʰ), represents the influence line for the quantity X_1. [The correction to be applied to X_2 by equ. (320), $\frac{b\,I_0}{A_0 . a_0 . r_0 \Sigma v''} . H = c . H$, can be introduced; in general, however, this quantity will be so very small that it may safely

Fig. 65.



be neglected.] If the pole distance for the force polygon (Fig. 65ᵇ) be chosen equal to $\frac{1}{n} \sum\limits_0^l v''$, then X_2 is to be measured on a scale whose unit $= n$ force-units. The scale for X_1 is given by $\sum\limits_0^l v'_m\, x_m$, i. e., by the distance $\overline{p\,q}$ which is intercepted on the Y-axis between the first and last sides of the funicular polygon. From the influence lines for X_1 and X_2 we may easily derive the curves for $(e_2 - e_1)$ and z_0 which are to be used directly in the construction of the end reactions. To do this, we simply multiply the ordinates of the X_1-curve by the ratios $\frac{G}{H} \cdot \frac{l}{pq}$, and those of the X_2-curve by the ratios

$\frac{G}{nH}$; the corresponding figures indicate how this may be accomplished graphically. The construction of the reaction lines no longer offers any difficulties, since the distances z_0 and $(e_2 - e_1)$ fix the closing side of the equilibrium polygon $D\,E\,F$, which is to be constructed with H as pole distance. If this polygon be drawn for different positions of the load, the curve generated by the vertex-point F will be the *reaction locus* and the enveloping curves of the reaction lines will be the *tangent curves.*

Through X_1 we also obtain the vertical end-reactions V_1 and V_2. These will be for supports at the same elevation:

$$\left.\begin{array}{l} V_1 = G\frac{l-\xi}{l} + \frac{M_2 - M_1}{l} = \mathbf{V}_1 + H\,.\,\frac{e_2 - e_1}{l} = \mathbf{V}_1 + X_1 \\ V_2 = \mathbf{V}_2 - X_1 \end{array}\right\} \dots\dots (322.$$

where $\mathbf{V}_1$ and $\mathbf{V}_2$ denote the reactions for a simple beam. Since X_1 is influenced only by the variation of the moment of inertia and not by the form of the arch, it follows that V_1 and V_2 are also independent of the arch-curve.

The end moments are given by the formulae:

$$\left.\begin{array}{l} M_1 = H\,.\,t_0 - X_1\frac{l}{2} - X_2 = H\,.\,e_1 \\ M_2 = H\,.\,t_0 + X_1\frac{l}{2} - X_2 = H\,.\,e_2 \end{array}\right\} \dots\dots\dots\dots (309^a.$$

b.) *Horizontal Loading.* If a concentration W is supposed to act on the arch horizontally, or parallel to the arch-chord (Fig. 65ª), at a point whose horizontal distance from the crown is w, there may be derived from the fundamental equations (307) the following equations of condition for the unknowns H_w, X_{1w} and X_{2w} which have the same significance as the similar quantities in the preceding analysis:

$$\left.\begin{array}{l} H_w = \dfrac{\Sigma \mathbf{M} v + W\,.\,\frac{I_0}{A_0}\sum\limits_w^{\frac{l}{2}} \frac{\Delta x}{A}}{\Sigma y v + \frac{I_0\, l}{a_0 A_0}} \\ X_{1w}.\,l\left(1 - \frac{1}{2}\frac{W}{H_w}\right) - X_{2w}\,.\,\frac{W}{H_w} \\ \qquad - \frac{\Sigma \mathbf{M} v'}{\Sigma x v'}\cdot l + W t_0 = 0 \\ X_{1w}\cdot\frac{l}{4}\frac{W}{H_w} - X_{2w}\left(1 - \frac{1}{2}\frac{W}{H_w}\right) \\ \qquad + \frac{\Sigma \mathbf{M} v''}{\Sigma v''} - \frac{1}{2} W t_0 + C = 0. \end{array}\right\} \dots\dots\dots (323.$$

Here **M** denotes the moments producible about the points of the arch-axis by the load W if the right end of the rib were free to slide horizontally; v, v' and v'' are the quantities defined by equs. (314) to (316); and $C = \frac{I_0}{a_0 r_0}\left[\frac{H_w l}{A_0} - W \sum_w^{\frac{l}{2}} \frac{\Delta x}{A}\right]\frac{1}{\Sigma v''}$ is a correction term which is so small that it may safely be neglected.

H_w is the horizontal (inward) thrust at the end B, and $(H_w - W)$ is the corresponding thrust at the end A.

The summations $\Sigma \mathbf{M} v$, $\Sigma \mathbf{M} v'$ and $\Sigma \mathbf{M} v''$ may be determined graphically as the moments producible about the line of action of the load, W, by forces v, v', v'' conceived as acting horizontally, together with opposing horizontal forces at the ends A and B equal in magnitude to the vertical reactions which would be produced if the forces v, v', v'' were applied vertically. Hence the summations are given by the intercepts of the funicular polygons Fig. 65^f, Fig. 65^1 and Fig. 65^c, being represented by the lengths $\alpha\beta$, $\gamma\delta$, and ϕ, respectively, measured by the same scale units as were used in the preceding construction for H, X_1 and X_2 for vertical loading.

The end moments for this case are given by:

$$\left.\begin{aligned} M_1 &= H_1 e_1 = \left(Ht_0 - X_1 \frac{l}{2} - X_2\right)\left(1 - \frac{W}{H}\right) \\ M_2 &= H_2 e_2 = Ht_0 + X_1 \frac{l}{2} - X_2 \end{aligned}\right\} \quad \ldots . \ (309^b.$$

2. Simplification with Constant Moment of Inertia, I'. As a rule the variation of the moment of inertia in hingeless arches is so slight, that we may assume $I' = I \cdot \cos\phi =$ constant. Then, with a uniform panel-length a, we have

$$\begin{cases} v_m = \frac{1}{6}\,(y_{m-1} + 4y_m + y_{m+1}) \\ v'_m = x_m \\ v''_m = 1. \end{cases}$$

Frequently, with greater subdivision into panels, we may also write $v_m = y_m$.

But, for a constant v'', the influence line for X_2 becomes a parabola; and, for a load G at a distance ξ from the end, equ. (313) becomes:

$$X_2 = \frac{1}{l}\int \mathbf{M}\,dx + C = G\,\frac{\xi(l-\xi)}{2l} + C$$

Here C denotes the small quantity $H\,\frac{I_0 b}{A_0 . l . r_0}$; if this is neglected, we obtain:

$$z_0=\frac{1}{2}\frac{G}{H}\cdot\frac{\xi(l-\xi)}{l}=\frac{1}{2}\overline{FH}\ldots\ldots\ldots\ldots\ldots\ldots(324.$$

i. e., z_0 is equal to one-half the altitude of the simple moment triangle DEF (Fig. 65[a]). Similarly, with the same assumptions as above, we find, from (312):

$$X_1=\frac{\int_{-\frac{l}{2}}^{+\frac{l}{2}}\mathbf{M}\,x\,dx}{\int_{-\frac{l}{2}}^{+\frac{l}{2}}x^2\,dx}=G\frac{\xi(l-\xi)}{l}\cdot\frac{(l-2\xi)}{l^2}=2X_2\frac{(l-2\xi)}{l^2}$$

or, referring back to equ. (308),

$$\frac{e_2-e_1}{2}=z_0\frac{l-2\xi}{l}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(325.$$

Accordingly z_0 may be derived from (e_2-e_1) by a simple construction.

Furthermore, by equ. (322),

$$\begin{cases}V_1=\mathbf{V}_1+X_1=G\dfrac{(l-\xi)^2(l+2\xi)}{l^3}\ldots\ldots\ldots(326.\\ V_2=\mathbf{V}_2-X_1=G\dfrac{\xi^2(3l-2\xi)}{l^3}\ldots\ldots\ldots\ldots(327.\end{cases}$$

The vertical reactions are thus identical with those occurring in a straight beam having fixed ends and a constant moment of inertia.

The ordinates of the reaction locus, measured from the axis A_x, are given by equ. (198) after substituting the values from (308) and (309[a]):

$$\eta=\frac{G}{H}\cdot\frac{\xi(l-\xi)}{l}-z_0-\frac{e_2-e_1}{l}\left(\frac{l}{2}-\xi\right);$$

substituting the values for z_0 and (e_2-e_1), this equation reduces to:

$$\eta=\frac{G}{H}\cdot\frac{2\xi^2(l-\xi)^2}{l^3}=8\frac{H}{G}\frac{z_0^2}{l}$$

The ordinate of the point D (Fig. 66) is also found:

$$z_1 = z_0 + \frac{c_2 - e_1}{2} = 2\,\frac{l-\xi}{l}\,z_0$$

or,

$$z_1 = \frac{\xi(l-\xi)^2}{l^3}\,\frac{G}{H} = \frac{1}{2}\frac{l}{\xi}\cdot\eta \quad \text{..................} \quad (328.$$

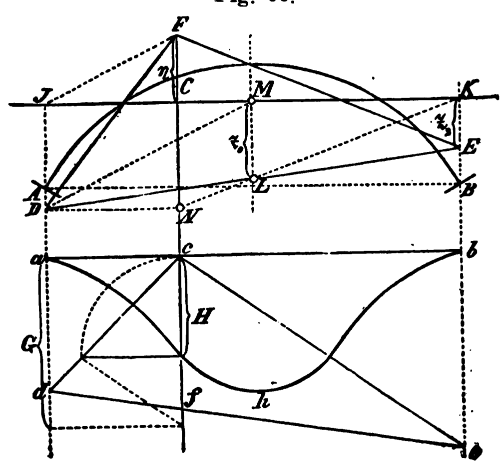
Hence the end moments will be given by:

$$M_1 = -H(z_1 - t_0) = -G\,\frac{\xi(l-\xi)^2}{l^2} + H\,t_0 \text{......} \quad (329.$$

and

$$M_2 = -H(z_2 - t_0) = -G\,\frac{\xi^2(l-\xi)}{l^2} + H\,t_0 \text{......} \quad (330.$$

Fig. 66.



In these expressions, the terms not containing H are identical with the end moments of a straight beam fixed at the ends.

On the basis of the foregoing results, the construction given in Figs. (65^a) to (65^k) for the general case may be simplified as follows for the special case of $I' =$ constant: First, by the method described previously (equ. 318) construct the H-curve $a\,h\,b$ (Fig. 66); also, with H as pole distance, for the given position of the load, construct the moment triangle $d\,c\,e$. Now lay off $\overline{ML} = z_0 = \frac{1}{2}\,\overline{cf}$, (equ. 324), draw the connecting line $\overline{KL}$ and the line $\overline{ND}$ parallel to the arch-chord. Then $\overline{DL}$ fixes the closing side of the equilibrium polygon (equ. 328);

and the vertex thereof may be obtained by joining D with M and drawing $\overline{JF}$ parallel to $\overline{DM}$ (equ. 328). As a check, the altitude of the point F above the closing line must equal the length $\overline{cf}$. We thus obtain the successive points F of the reaction locus as well as the tangents $\overline{DF}$ and $\overline{FE}$ to the tangent curve.

Assuming a constant moment of inertia I' and neglecting the effect of axial compression, the equations of condition (307) may be interpreted as follows: 1.) The total area between the axis of the arch and the line of pressures must vanish. $\left(\int M dx = 0.\right)$ 2.) If we conceive the elements of the arch-axis to be invested with weights proportional to the elements of the above-described area, affixing different signs to the areas on the two sides of the arch-axis, then the static moment about any two axes must vanish. $\left(\int M\,y\,dx = 0 \text{ and} \int M\,x\,dx = 0\right)$; in other words, the two areas, included between any horizontal line and the arch-axis and pressure-line respectively, must have a common center of gravity. *Winkler** has combined the above conditions in a single theorem: "In an arch of uniform cross-section, that line of pressures is approximately the right one for which the sum of the squares of the deviations from the arch-axis is a minimum." This theorem may be derived from the principle of least work, whereby $W = \int \frac{M^2}{EI'}\,dx + \int \frac{P^2}{EA}\,ds = min.$, or, with the above approximations, $\int M^2 dx = min.$

3. Flat Parabolic Arch of Constant Moment of Inertia. The rectifying line of the arch-curve is, in this case, located at $\frac{2}{3}$ the rise of the arch. For a load G placed at a distance ξ from the end, equ. (311), with the values $y = \frac{4f}{l^2}\chi\,(l - x) - \frac{2}{3}f$ and $I' =$ constant, will yield the following expression:

$$H = \frac{15}{4}\frac{\xi^2 (l-\xi)^2}{f\,l^3} \cdot \frac{G}{1 + \frac{45}{4}\frac{I}{A_0 f^2}} \quad \ldots\ldots\ldots\ldots (331.$$

For the vertical reactions V_1 and V_2, the expressions of (326) and (327) remain valid. The equation of the reaction locus becomes,

$$\eta = \frac{8}{15} f \left(1 + \frac{45}{4}\frac{I}{A_0 f^2}\right); \quad \ldots\ldots\ldots\ldots (332.$$

hence, in the present case, the reaction locus is a straight line

* Deutsche Bauzeitung, 1879, p. 128.

parallel to the axis of abscissae. The directions of the reaction lines are then fixed by the values, from (328), of $z_1 = \frac{1}{2}\frac{l}{\xi}\eta$ and $z_2 = \frac{1}{2}\frac{l}{(l-\xi)}\cdot\eta$; they are most conveniently obtained by the construction shown in Fig. 66. In addition, we may also establish the equations of the tangent curves which, for this case, are hyperbolas.

This is done with the aid of the following diagram (Fig. 67). If x_r and y_r are the coordinates of any point of the line of action of the left reaction referred to the point E as origin, then the equation of this reaction line is

$$y_r = \frac{\eta + z_1}{\xi} x_r - z_1 = \eta\left(\frac{2\xi + l}{2\xi^2}\cdot x_r - \frac{l}{2\xi}\right) \quad \text{(333.}$$

or

$$2\xi^2 y_r = \xi\eta\,(2x_r - l) + l\eta x_r \quad (\alpha.$$

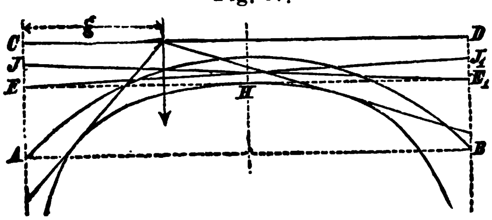
Differentiating this with respect to ξ,

$$4\xi y_r = \eta\,(2x_r - l) \quad (\beta.$$

Eliminating ξ from (α) and (β), we obtain the following equation of the tangent curve:

$$(2x_r - l)^2\eta + 8x_r y_r l = 0 \quad (333^a.$$

Fig. 67.



$A\,E = B\,E_1 = \frac{2}{3}f$

$E\,C = E_1 D = \eta = \frac{8}{15}f\left(1 + \frac{45}{4}\frac{I_0}{A_0 f^2}\right)$

$E\,J = J\,C = \frac{1}{2}\eta$

Of the two asymptotes of the hyperbola represented by the above equation (333^a), one coincides with the vertical line through the end of the arch-axis, and the other is an inclined line ($J\,E_1$) passing through the points $\left(x=0, y=\frac{1}{2}\eta\right)$ and $(x=l, y=0)$. At $x = \frac{l}{2}$, or the center of the span, the hyperbola is tangent to the axis of abscissae.

The end moments are

$$\left.\begin{aligned} M_1 &= -H\left(\frac{1}{2}\frac{l}{\xi}\eta - \frac{2}{3}f\right) \\ &= -G\frac{\xi\,(l-\xi)^2}{2\,l^3}\left\{2\,l - \frac{5\,\xi}{1 + \frac{45}{4}\frac{I}{A_0 f^2}}\right\} \end{aligned}\right\} \quad \text{(334.}$$

$$\left.\begin{aligned} M_2 &= -H\left(\frac{1}{2}\,\frac{l}{(l-\xi)}\,\eta-\frac{2}{3}\,f\right)\\ &= -G\,\frac{\xi^2\,(l-\xi)}{2\,l^3}\left\{2\,l-\frac{5\,(l-\xi)}{1+\frac{45}{4}\,\frac{I}{A_0 f^2}}\right\}\end{aligned}\right\}\quad\ldots\ldots(335.$$

Neglecting the effect of the axial compression, we may substitute in all the above equations $1+\frac{45}{4}\,\frac{I}{A_0 f^2}=1$ and $\eta=\frac{8}{15}\,f$.

4. Effect of a Change of Temperature of the Arch and of a Displacement of the Abutments. Let the arch, supposed weightless and without load, alter its temperature uniformly in all parts by t°; also let the span be increased, through a displacement of the abutments, by an amount $\Delta\,l$. Then, retaining the assumptions introduced at the beginning of this article, the fundamental equations derived from equs. (185) and (186) take the following form:

$$\left.\begin{aligned} &\int_{-\frac{l}{2}}^{+\frac{l}{2}}\frac{M}{I'}\,dx+(H-E\,A_0\,\omega\,t)\,\frac{b}{A_0 r_0}=0\\ &\int_{-\frac{l}{2}}^{+\frac{l}{2}}\frac{M}{I'}\,y\,dx-(H-E\,A_0\,\omega\,t)\,\frac{l}{A_0}=E\,.\Delta\,l\\ &\int_{-\frac{l}{2}}^{+\frac{l}{2}}\frac{M}{I'}\,x\,dx=0\end{aligned}\right\}\quad\ldots\ldots(336.$$

Also, with $\mathbf{M}=0$, equ. (309) becomes

$$M=-Hy-X_1\,x-X_2$$

Choosing the axis of abscissae, as before, so that $\int_{-\frac{l}{2}}^{+\frac{l}{2}}\frac{y\,dx}{I'}=0$,

the solution of the above equations will yield the following expressions:

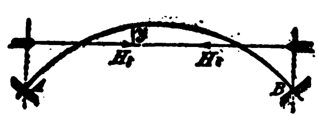
$$\left\{\begin{array}{l} H_t = \dfrac{\dfrac{E(\omega t l - \Delta l)}{l}}{\displaystyle\int_0^l \frac{y^2 d x}{I'} + \frac{l}{A_0}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (337. \\ X_1 = 0 \\ X_2 = \dfrac{(H - E A_0 \omega t)\, b}{A_0 r_0 \displaystyle\int_0^l \frac{d x}{I'}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (338. \end{array}\right.$$

Introducing the quantities v, v', v'', defined by equs. (314) to (316), we obtain

$$\left\{\begin{array}{l} H_t = \dfrac{E I_0 (\omega t l - \Delta l)}{a_0 \left(\displaystyle\sum_0^l y_m v_m + \frac{I_0 l}{A_0 a_0}\right)} \quad \ldots\ldots (339. \\ X_1 = 0, \quad X_2 = \dfrac{I_0 b (H - E A_0 \omega t)}{A_0 a_0 r_0 \displaystyle\sum_0^l v''_m} \quad \ldots\ldots\ldots (340. \end{array}\right.$$

The denominators of the above expressions are determined by the previously described graphic construction. X_2 will always be a vanishing quantity and may, without sensible error, be equated to zero. The line of action of the horizontal thrust thus coincides with the X-axis, i. e., if $I' =$ constant, with the rectifying line of the arch-curve (Fig. 68).

Fig. 68.



With parabolic arch curve and constant I', since $\int_0^l y^2\, d x = \frac{4}{45} f^2 l$,

$$H_t = \frac{45}{4 f^2} \frac{E I_0 \left(\omega t - \frac{\Delta l}{l}\right)}{1 + \frac{45}{4} \frac{I_0}{A_0 f^2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (341.$$

or, introducing the constant ordinate, η, of the reaction locus as given by equ. (332),

$$H_t = \frac{6 E I_0}{\eta f} \left(\omega t - \frac{\Delta l}{l}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (342.$$

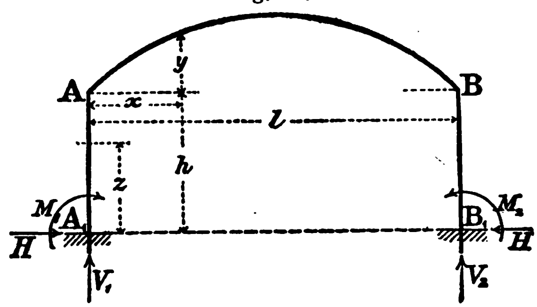
Comparing this with the corresponding (approximate) expression for the parabolic arch with *hinged ends*, as given by equ. (290^a):

$$H'_t = \frac{15}{8 f^2} \frac{E I_0 \left(\omega t - \frac{\Delta l}{l}\right)}{1 + \frac{15}{8} \frac{I_0}{A_0 f^2}},$$

it appears that in the hingeless arch the horizontal thrust due to temperature change or displacement of the abutments is nearly six times as great as the corresponding thrust produced in the two-hinged arch under the same conditions.

5. Arch Connected to Elastic Piers. Let the arch-axis AB (Fig. 69) be continued in the vertical pier-axes AA_1 and BB_1. We assume that the two piers, throughout their height h, have constant moments of inertia I_1 and are formed of a material whose coefficient of elasticity is $E_1 = \epsilon . E$. The base-sections, A_1 and B_1, are assumed incapable of rotation or displacement. Let the arch-rib have a variable moment of inertia I and a mean cross-section A_0. In calculating the deformations, we will consider the axial force in the arch constant and $= H$, and neglect the effect of the corresponding force in the piers.

Fig. 69.



With the notation indicated in Fig. 69 or previously introduced, we have, for a vertical loading on the arch, $V_1 = \mathbf{V}_1 + X_1$,

at any section of the left pier, $M = M_1 - Hz$
at any section of the right pier, $M = M_1 - Hz + X_1 l$
at any section of the arch, $M = M_1 + X_1 x - H(h + y) + \mathbf{M}$

Substituting the following symbols for certain definite integrals:

$$a = \int_0^l \frac{ds}{I}, \quad b = \int_0^l \frac{y\,ds}{I}, \quad c = \int_0^l \frac{y^2 ds}{I}, \quad d = \int_0^l \frac{x^2 ds}{I} \quad \ldots\ldots (343.$$

and equating to zero the differential coefficients of the work of deformation with respect to the three unknowns H, X_1, M_1, there are obtained the following three equations of condition:

$$\left.\begin{aligned}
-E\frac{dW}{dH} &= M_1\left(ah+b+\frac{h^2}{\epsilon I_1}\right)-H\Big(ah^2+2hb+c\\
&\quad+\frac{2}{3}\frac{h^3}{\epsilon I_1}+\frac{l}{A_0}\Big)+\frac{1}{2}X_1\left(ah+b+\frac{h^2}{\epsilon I_1}\right)l\\
&\quad+h\int_0^l\frac{\mathbf{M}\,ds}{I}+\int_0^l\frac{\mathbf{M}y}{I}\,ds=0\\
+E\frac{dW}{dM_1} &= M_1\left(a+\frac{2h}{\epsilon I_1}\right)-H\left(ah+b+\frac{h^2}{\epsilon I_1}\right)\\
&\quad+X_1\left(a+\frac{2h}{\epsilon I_1}\right)\frac{l}{2}+\int_0^l\frac{\mathbf{M}}{I}\,ds=0\\
+E\frac{dW}{dX_1} &= M_1\left(a+\frac{2h}{\epsilon I_1}\right)\frac{l}{2}-H\left(ah+b+\frac{h^2}{\epsilon I_1}\right)\frac{l}{2}\\
&\quad+X_1\left(d+\frac{hl^2}{\epsilon I_1}\right)+\int_0^l\frac{\mathbf{M}x}{I}\,ds=0
\end{aligned}\right\} \quad \text{. . (344.}$$

Solving these equations, we obtain

$$H=\frac{\left(a+2\frac{h}{\epsilon I_1}\right)\int_0^l\frac{\mathbf{M}y}{I}\,ds-\left(b-\frac{h^2}{\epsilon I_1}\right)\int_0^l\frac{\mathbf{M}}{I}\,ds}{ac-b^2+\frac{2h}{\epsilon I_1}\left(c+bh+\frac{ah^2}{3}+\frac{h^3}{6\epsilon I_1}\right)+\left(a+2\frac{h}{\epsilon I_1}\right)\frac{l}{A_0}} \quad \text{. . (345.}$$

and

$$X_1=\frac{\int_0^l\frac{\mathbf{M}x}{I}\,ds-\frac{l}{2}\int_0^l\frac{\mathbf{M}}{I}\,ds}{\frac{1}{4}l^2a-d-l^2\frac{h}{\epsilon I_1}}; \quad \text{. . . . . (346.}$$

and M_1 may be found by substituting these values in any one of equations (344).

If $h=0$, the above equations reduce to

$$H=\frac{a\int_0^l\frac{\mathbf{M}y}{I}\,ds-b\int_0^l\frac{\mathbf{M}}{I}\,ds}{ac-b^2+a\frac{l}{A_0}} \quad \text{. . . . . (347.}$$

$$X_1=\frac{\int_0^l\frac{\mathbf{M}x}{I}\,ds-\frac{l}{2}\int_0^l\frac{\mathbf{M}}{I}\,ds}{\frac{1}{4}l^2a-d} \quad \text{. . . . . (348.}$$

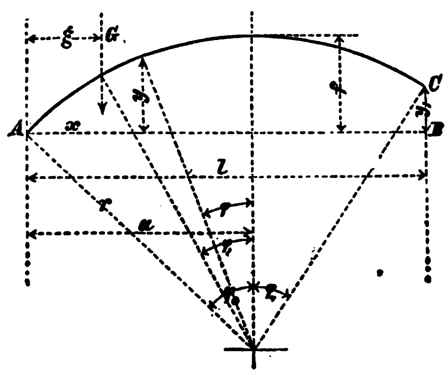
and

$$M_1 a - H b + \frac{1}{2} l a X_1 + \int \frac{M}{I} ds = 0 \ldots\ldots\ldots (349.$$

This constitutes another solution for the general case (1.) as previously given by equs. (311) to (313).

6. Segmental Arch with Constant Moment of Inertia. It is usually advisable to adhere to the general formulae established in paragraph (1) of this article for any form of arch. Nevertheless, for segmental arches, it is desirable to present here a more exact analysis* to be applied whenever the sectional dimensions cannot, as above, be considered negligibly small in comparison with the radius of curvature.

Fig. 70.



With the notation of Fig. 70, the fundamental equations derived from equs. (182), (185) and (186) may be written as follows:

$$\left\{\begin{array}{l} \left.\begin{array}{l}\Delta\phi - \Delta\phi_0 = -r \displaystyle\int_{\phi_0}^{\phi} \left(\frac{P}{EAr} + \frac{M}{EAr^2} + \frac{M}{EJ}\right) d\phi \\ \qquad + \omega t (\phi - \phi_0)\end{array}\right\} \ldots\ldots\ldots (350. \\ \Delta x - \Delta x_0 = -y . \Delta\phi_0 - r\cos\phi(\Delta\phi - \Delta\phi_0) - r^2 \displaystyle\int_{\phi_0}^{\phi} \frac{M\cos\phi\, d\phi}{EJ} \ldots (351. \\ \Delta y - \Delta y_0 = x . \Delta\phi_0 - r\sin\phi(\Delta\phi - \Delta\phi_0) - r^2 \displaystyle\int_{\phi_0}^{\phi} \frac{M\sin\phi\, d\phi}{EJ} \ldots\ldots (352. \end{array}\right.$$

Assuming a constant area of cross-section and writing

$$\frac{J}{Ar^2} = \delta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (353.$$

*Schäffer—"Handbuch des Brückenbaues"—Chapter XI—1st edition.

and observing that, for vertical loads,

$$P = \left(V_1 - \sum_0^x G\right) \sin\phi + H\cos\phi \qquad (354.$$

and

$$\left.\begin{aligned} M = M_1 + V_1 r\,(\sin\phi_0 - \sin\phi) - H\,r\,(\cos\phi - \cos\phi_0) \\ - \sum_0^x G\,r\,(\sin\phi_1 - \sin\phi) \end{aligned}\right\} \qquad (355.$$

where M_1, H and V_1 refer to the left end, we obtain:

$$\left.\begin{aligned} \Delta\phi - \Delta\phi_0 &= -r^2 \int_{\phi_0}^{\phi} \frac{\delta P d\phi}{EJ} - r\int_{\phi_0}^{\phi} \frac{(1+\delta)\,M.d\phi}{EJ} + \omega t(\phi - \phi_0) \\ &= -\frac{r}{EJ}\Big[M_1(1+\delta)(\phi-\phi_0) + V_1\Big\{r(\cos\phi - \cos\phi_0) \\ &\qquad + (1+\delta)\,r\,(\phi-\phi_0)\sin\phi_0\Big\} \\ &\quad + H\Big\{-r(\sin\phi - \sin\phi_0) + (1+\delta)\,r\,(\phi-\phi_0)\cos\phi_0\Big\} \\ &\quad + \sum_0^x G\Big\{-r(\cos\phi - \cos\phi_1) - (1+\delta)\,r\,(\phi-\phi_1)\sin\phi_1\Big\}\Big] \\ &\qquad + \omega t(\phi - \phi_0) \\ &= -\frac{r}{EJ}\Big[M_1(1+\delta)(\phi-\phi_0) + V_1\Big\{y + (1+\delta)\,a\,(\phi-\phi_0)\Big\} \\ &\qquad + H\Big\{x + (1+\delta)(r-f)(\phi-\phi_0)\Big\} \\ &\quad - \sum_0^x G\Big\{y - h + (1+\delta)(a-\xi)(\phi-\phi_1)\Big\}\Big] + \omega t(\phi-\phi_0) \end{aligned}\right\} \qquad (356.$$

$$\left.\begin{aligned} \Delta x - \Delta x_0 &= -y\,\Delta\phi_0 - r\cos\phi\,(\Delta\phi - \Delta\phi_0) - r^2\int_{\phi_0}^{\phi} \frac{M\cos\phi\,d\phi}{EJ} \\ &= -y\,\Delta\phi_0 - r\cos\phi\,(\Delta\phi - \Delta\phi_0) - \frac{r^2}{EJ}\Big[M_1(\sin\phi - \sin\phi_0) \\ &\quad - \frac{V_1 r(\sin\phi - \sin\phi_0)^2}{2} - H r\Big\{\frac{\phi-\phi_0}{2} + \frac{\sin\phi}{2}(\cos\phi - \cos\phi_0) \\ &\qquad - \frac{\cos\phi_0}{2}(\sin\phi - \sin\phi_0)\Big\} + \sum_0^x \frac{G r}{2}(\sin\phi_1 - \sin\phi)^2\Big] \\ &= -y.\Delta\phi_0 - r\cos\phi\,(\Delta\phi - \Delta\phi_0) - \frac{r}{EJ}\Big[-M_1 x - \frac{V_1 x^2}{2} \\ &\quad - H\left(\frac{r^2(\phi-\phi_0)}{2} + \frac{(a-x)\,y}{2} + \frac{(r-f)\,x}{2}\right) + \sum_0^x \frac{G(x-\xi)^2}{2}\Big], \end{aligned}\right\} \qquad (357.$$

where $(\Delta\phi - \Delta\phi_0)$ is to be found by equ. (356).

$$\left.\begin{aligned}
&\Delta y-\Delta y_0=x\,.\,\Delta\phi_0-r\sin\phi\,(\Delta\phi-\Delta\phi_0)-r^2\int_{\phi_0}^{\phi}\frac{M\sin\phi\,d\phi}{EJ}\\
&=x\,.\,\Delta\phi_0-r\sin\phi\,(\Delta\phi-\Delta\phi_0)-\frac{r^2}{EJ}\Big[-M_1\,(\cos\phi-\cos\phi_0)\\
&-V_1 r\left\{\frac{\phi-\phi_0}{2}-\frac{\cos\phi}{2}(\sin\phi-\sin\phi_0)-\frac{\sin\phi_0}{2}(\cos\phi_0-\cos\phi)\right\}\\
&+\frac{Hr}{2}(\cos\phi-\cos\phi_0)^2+\sum_0^x Gr\left\{\frac{\phi-\phi_1}{2}+\frac{\sin\phi_1}{2}(\cos\phi-\cos\phi_1)\right.\\
&\left.+\frac{\cos\phi}{2}(\sin\phi_1-\sin\phi)\right\}\Big]=x\Delta\phi_0-(a-x)\,(\Delta\phi-\Delta\phi_0)\\
&-\frac{r}{EJ}\Big[-M_1 y+V_1\left\{\frac{r^2(\phi-\phi_0)}{2}+\frac{x\,(y+r-f)}{2}+\frac{ay}{2}\right\}+\frac{Hy^2}{2}\\
&+\sum_0^x G\left\{\frac{r^2(\phi-\phi_1)}{2}+\frac{(a-\xi)\,(y-h)}{2}+\frac{(y+r-f)\,(x-\xi)}{2}\right\}\Big]
\end{aligned}\right\}\quad\text{.. (358.}$$

Here h is the ordinate of the arch-curve corresponding to the angle ϕ_1.

If we substitute $x=l$, $y=y_1$, $\phi=\phi_{,,}$, the above equations (356), (357); (358) will constitute the three required equations of condition; these need not be rewritten since they do not differ from the foregoing except in the affixing of the subscripts and the substitution of l for x.

If we have a *symmetrical segmental arch,* then

$$a=\frac{l}{2},\quad y_1=0,\quad \phi_{,,}=-\phi_0;$$

and if we write, for the ends of the span, $\Delta\phi-\Delta\phi_0=c_1$, $\Delta x-\Delta x_0=c_2$, $\Delta y-\Delta y_0=c_3$, we obtain the following expressions for the equations of condition:

$$\left.\begin{aligned}
\Delta\phi-\Delta\phi_0=c_1=-\frac{r}{EJ}\Big[&-2M_1\,(1+\delta)\,\phi_0-V_1\,(1+\delta)\,l\,\phi_0\\
&+2H\left\{\frac{l}{2}-(1+\delta)\,(r-f)\,\phi_0\right\}\\
&+\sum_0^l G\left\{h+\frac{(1+\delta)\,(l-2\xi)\,(\phi_0+\phi_1)}{2}\right\}\Big]-2\,\omega\,t\,\phi_0
\end{aligned}\right\}\quad\text{.. (359.}$$

$$\left.\begin{aligned}
\Delta x-\Delta x_0=c_2=-c_1\,r\cos\phi_0-\frac{r}{EJ}\Big[&-M_1 l+\frac{V_1 l^2}{2}\\
&+H\left\{r^2\phi_0-\frac{l\,(r-f)}{2}\right\}+\sum_0^l\frac{G\,(l-\xi)^2}{2}\Big]
\end{aligned}\right\}\quad\text{.. (360.}$$

$$\left.\begin{aligned}\Delta y_1-\Delta y_0=c_3=l.\Delta\phi_0+\frac{c_1 l}{2}-\frac{r}{EJ}\Bigg[V_1\left\{r^2\phi_0-\frac{l(r-f)}{2}\right\}\\ -\sum_0^l\frac{G}{2}\left\{r^2(\phi_0+\phi_1)+\frac{h(l-2\xi)}{2}-(r-f)(l-\xi)\right\}\Bigg]\end{aligned}\right\}\quad ..(361.$$

Equ. (361) yields:

$$\left.\begin{aligned}V_1=&-\frac{EJ(c_3-l\Delta\phi_0-c_1 r\sin\phi_0)}{r^3(\phi_0-\sin\phi_0\cos\phi_0)}\\ &+\sum_0^l G\frac{(\phi_0+\phi_1-\sin\phi_0\cos\phi_0-2\sin\phi_1\cos\phi_0+\sin\phi_1\cos\phi_1)}{2(\phi_0-\sin\phi_0\cos\phi_0)}\\ =&-\frac{EJ(2c_3-2l\Delta\phi_0-c_1 l)}{r[2r^2\phi_0-l(r-f)]}\\ &+\sum_0^l G\frac{2r^2(\phi_0+\phi_1)+h(l-2\xi)-2(r-f)(l-\xi)}{2r^2\phi_0-l(r-f)}\end{aligned}\right\}\quad ..(362.$$

From equs. (359) and (360) we obtain

$$\left.\begin{aligned}H=&\frac{EJ[-(1+\delta)\phi_0(c_2+c_1 r\cos\phi_0)+c_1 r\sin\phi_0]}{r^3[\phi_0(1+\delta)(\phi_0+\sin\phi_0\cos\phi_0)-2\sin^2\phi_0]}\\ &+\frac{2EJ\omega t\phi_0\sin\phi_0}{r^3[\phi_0(1+\delta)(\phi_0+\sin\phi_0\cos\phi_0)-2\sin^2\phi_0]}\\ &+\sum_0^l G\frac{2\sin\phi_0[\cos\phi_1-\cos\phi_0+\phi_1(1+\delta)\sin\phi_1]-(1+\delta)\phi_0(\sin^2\phi_0+\sin^2\phi_1)}{2[\phi_0(1+\delta)(\phi_0+\sin\phi_0\cos\phi_0)-2\sin^2\phi_0]}\\ =&\frac{EJ[-2(1+\delta)\phi_0(c_2+c_1(r-f))+c_1 l]}{r[\phi_0(1+\delta)(2r^2\phi_0+l(r-f))-l^2]}\\ &+\frac{2EJ\omega t\phi_0 l}{r[\phi_0(1+\delta)(2r^2\phi_0+l(r-f))-l^2]}\\ &+\sum_0^l G\frac{l[2h+\phi_1(1+\delta)(l-2\xi)]-(1+\delta)\phi_0(l^2-2l\xi+2\xi^2)}{2[\phi_0(1+\delta)(2r^2\phi_0+l(r-f))-l^2]}\end{aligned}\right\}\quad ..(363.$$

With these values, M_1 may be calculated by either equ. (359) or (360); the latter yields:

$$\left.\begin{aligned}M_1=&\frac{EJ(c_2+c_1 r\cos\phi_0)}{2r^2\sin\phi_0}-V_1 r\sin\phi_0\\ &+\frac{H(\phi_0-\sin\phi_0\cos\phi_0)r}{2\sin\phi_0}+\sum_0^l\frac{Gr(\sin\phi_0+\sin\phi_1)^2}{4\sin\phi_0}\\ =&\frac{EJ(c_2+c_1(r-f))}{rl}-\frac{V_1 l}{2}+\frac{H(2r^2\phi_0-l(r-f))}{2l}\\ &+\sum_0^l\frac{G(l-\xi)^2}{2l}\end{aligned}\right\}\quad ..(364.$$

M_1 is most readily found in this manner after V_1 and H have been determined. Then we also find

$$-V_2 = V_1 - \sum_0^l G \quad \text{(365.}$$

$$M_2 = M_1 + V_1 l - \sum_0^l G (l - \xi) \quad \text{(366.}$$

The preceding equations express not only the effect of a concentrated load, but also that of a change of temperature; the latter effect, by itself, is obtained by putting $G = 0$ in the various formulae. Similarly, these formulae give the effects of a rotation or displacement of the abutments, if the magnitude of such displacement (c_1, c_2 or c_3) be known. If the ends are rigidly fixed, we must take $c_1 = c_2 = c_3 = 0$.

For a flat segmental arch, the formulae of the parabolic arch may be applied without sensible error.

§ 21. Maximum Moments and Shears.

In determining the critical loads for maximum moments and shears, we need merely to apply the general principles presented in § 14. The same general rules apply to the hingeless arch as to the arch with hinged ends except that the reaction lines do not pass through the end-hinges but are to be drawn tangent to the tangent curves.

Again, the maximum moments and shears may be determined either by the method of *influence lines* or by means of the *equilibrium polygon of the loading.* The influence lines for moments and shears, however, are not so simply found as for the hinged arch, but may be derived from the influence lines for H, X_1, X_2, according to the formulae (309), (192), (322), which may be rewritten:

$$\begin{cases} M = \mathbf{M} - Hy - X_1 x - X_2 & \text{.............. (367.} \\ S = (\mathbf{V}_1 + X_1) \cos\phi - \mathrm{H} \sin\phi & \text{.......... (368.} \end{cases}$$

where $\mathbf{M} = G \dfrac{(l-\xi)\ (l-2x)}{2l}$ or $G \dfrac{\xi\,(l+2x)}{2l}$, $V_1 = G \dfrac{l-\xi}{l}$, and x is measured from the center of the span. To find the extreme fiber stresses in the sections, the moments should be taken about the core-points. The method of influence lines is advantageous when the load consists of a train of concentrations.

To obtain the maximum moments and shears directly from the equilibrium polygon of the critical loading, it is necessary to find the values of the horizontal thrust, end reaction and end moment for that loading. It is a simple matter to obtain these quantities by summation of the values found for the different positions of a concentrated load; this may also be accomplished graphically. If, with the resulting values of H, V_1 and M_1, there is constructed the equilibrium polygon of the loading (or line of pressures) in its proper position with respect to the axis of the arch, then the moment at the section under investigation will be $M = H\,y'$, where y' is the distance of the equilibrium polygon above the arch-axis or, rather, above the core-point of the section. If the load is taken as uniformly distributed, we may use, as before, a second reaction locus and tangent curve to facilitate the construction of the equilibrium polygon.

For special arch forms, the influence of a partial uniform load may be calculated as follows:

1. **Any Arch-Curve, with $I' = $ Constant.** If the load of p per unit length extends for a distance λ from the left end, the resulting vertical end reactions may be derived from equs. (326) and (327) by integrating between the limits 0 and λ, giving

$$\begin{cases} V_1 = \dfrac{p\lambda\,(2l^3 - 2l\lambda^2 + \lambda^3)}{2l^3} \quad\ldots\ldots\ldots\ldots\ldots (369. \\[2ex] V_2 = \dfrac{p\lambda^3\,(2l - \lambda)}{2l^3} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (370. \end{cases}$$

If **H** is the horizontal thrust produced by this loading, the integration of the expressions of (329) and (330) yields the following values of the end moments:

$$\begin{cases} M_1 = -\left[p\,\lambda^2\,\dfrac{6l^2 - 8\lambda l + 3\lambda^2}{12\,l^2} - \mathbf{H}\,t_0\right] \ldots\ldots (371. \\[2ex] M_2 = -\left[p\,\lambda^2\,\dfrac{4l - 3\lambda}{12\,l^2} - \mathbf{H}\,t_0\right] \ldots\ldots\ldots (372. \end{cases}$$

where t_0 is defined by equ. (321).

For a full-span load, if $\mathbf{H}_{\text{tot}}$ denotes the corresponding horizontal thrust, we have

$$\begin{cases} V_1 = V_2 = \dfrac{1}{2}\,pl \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (373. \\[2ex] M_1 = M_2 = -\left[\dfrac{1}{12}\,pl^2 - \mathbf{H}_{\text{tot}}\,.\,t_0\right] \ldots\ldots\ldots (374. \end{cases}$$

If λ_1 is the distance to be covered with load, as determined by the rules of § 14, in order to produce the greatest positive moment at the core-point (x_k, y_k) (referred to the end A as origin), that moment will be given by

$$M_{\max} = M_2 + V_2\,(l - x_k) - \mathbf{H}y_k - p\,\frac{(\lambda_1 - x_k)^2}{2}.$$

$$\therefore\ M_{\max} = \frac{p\lambda^3}{12\,l^2}\left[8l - 3\lambda_1 - 6\,(2l - \lambda_1)\,\frac{x_k}{l}\right] - \mathbf{H}\,(y_k - t_0) - p\,\frac{(\lambda_1 - x_k)^2}{2} \ldots (375.$$

For a full load, there results

$$M_{\text{tot}} = \frac{1}{12}\,p\left[6x_k\,(l - x_k) - l^2\right] - \mathbf{H}_{\text{tot}}\,(y_k - t_0) \ldots\ldots\ldots (376.$$

with which we may calculate

$$M_{\min} = M_{\text{tot}} - M_{\max}.$$

If λ_2 is the abscissa of the critical point for shears, and if $\mathbf{H}_{x-\lambda_2}$ is the horizontal thrust for a load extending from

x to λ_2, we obtain, by the use of equs. (192) and (369), the following expressions for the greatest shears:

$$S_{\max}=\frac{p}{2l^3}\Big[(l-x)^2(l^2-x^2)-(l-\lambda_2)^2(l^2-\lambda_2^2)\Big]\cos\phi-\mathrm{H}_{x-\lambda_2}\sin\phi \quad (377.$$

$$S_{\min}=\frac{1}{2}p(l-2x)\cos\phi-\mathrm{H}_{\mathrm{tot}}\sin\phi-S_{\max} \quad (378.$$

2. Flat Parabolic Arch. If a uniformly distributed load p extends for a distance λ from one end of the span, the integration of equ. (331) yields

$$\mathrm{H}=\frac{p\lambda^3(10l^2-15l\lambda+6\lambda^2)}{8fl^3\left(1+\frac{45}{4}\frac{I_0}{A_0f^2}\right)}=\frac{5}{4}\left(\frac{\lambda}{l}\right)^3\left(1-\frac{3}{2}\frac{\lambda}{l}+\frac{3}{5}\frac{\lambda^2}{l^2}\right)\frac{pl^2}{f'} \quad (379.$$

where, as an abbreviation, $f'=f\left(1+\frac{45}{4}\frac{I_0}{A_0f^2}\right)$.

For a full-span load,

$$\mathrm{H}_{\mathrm{tot}}=\frac{pl^2}{8f}\cdot\frac{1}{1+\frac{45}{4}\frac{I_0}{A_0f^2}}=\frac{pl^2}{8f'} \quad (380.$$

The vertical reactions and the end moments M_1 and M_2 are to be determined by substituting the above values of $\mathbf{H}$ in the formulae previously developed; similarly, the maximum moments and shears may be found by equs. (375) and (377), when the distances λ_1 and λ_2, corresponding to the critical loads, are known. In the case of the parabolic arch, those distances may be calculated as follows:

For the maximum moment at the core-point (x_k, y_k), (referring to the end A as origin), we have, by equ. (333),

$$y_k-\frac{2}{3}f=\left(\frac{l+2\lambda_1}{2\lambda_1^2}x_k-\frac{l}{2\lambda_1}\right)\frac{8}{15}f\left(1+\frac{45}{4}\frac{I}{A_0f^2}\right)$$

or

$$\left(y_k-\frac{2}{3}f\right)\lambda_1^2+\frac{4}{15}f'\left(l-2x_k\right)\lambda_1=\frac{4}{15}f'lx_k \quad (381.$$

The critical length, λ_1, will equal l for any arch-point whose coordinates (x'_k, y'_k) satisfy the equation:

$$\left(y'_k-\frac{2}{3}f\right)l^2+\frac{4}{15}f'l^2=\frac{12}{15}f'lx'_k \quad (382.$$

If this yields $x'_k>\frac{l}{2}$, then, for all points in the middle portion between the abscissae $(l-x'_k)$ and (x'_k), the critical loading will be an interrupted one (with two critical points); and, for any of these points, we must add to the value of $M_{\max}$ given

by equ. (375) the value given by the same formula for the symmetrically located point of the arch.

The critical point for shears is determined by

$$\frac{2\lambda_2+l}{2\lambda_2^2}\cdot\frac{8}{15}f'=tan\,\phi=\frac{4f}{l^2}(l-2x)\ \ldots\ldots\ (383.$$

For a full-span load, the end moments will be, by equs. (374) and (380),

$$M_1=M_2=-\frac{1}{12}pl^2\left(1-\frac{f}{f'}\right)$$

and the crown moment will be, by (376) and (380),

$$M_c=\frac{1}{24}pl^2\left(1-\frac{f}{f'}\right).$$

If we put $f'=f$, which is equivalent to neglecting the effect of the axial force upon the deformation, we obtain, for a full load, $M_1=M_2=M_c=0$ so that the pressure line coincides with the axis of the arch. With respect to the error resulting from this approximation, practically the same remarks apply as in the case of the two-hinged arch (see page 137). If the arch-rib consists of two parallel flanges of combined area A with an effective depth between them of h, and if σ is the fiber stress for the pressure line coinciding with the arch-axis, then the extreme fiber stresses at the crown in the case of the eccentric location of the pressure line will be by equ. (179), in the upper flange,

$$\sigma_u=\sigma\frac{1+\frac{15}{8}\frac{h}{f}}{1+\frac{45}{16}\frac{h^2}{f^2}}$$

and, in the lower flange,

$$\sigma_l=\sigma\frac{1-\frac{15}{8}\frac{h}{f}}{1+\frac{45}{16}\frac{h^2}{f^2}}$$

The maximum value of the upper fiber stress occurs with $\frac{h}{f}=\frac{4}{15}$ and amounts to $\sigma_u=1.25\,\sigma$; while the lower fiber stress changes its sign, hence becomes a tensile stress, with $\frac{h}{f}\gtreqless\frac{8}{15}$.

Similarly we find, for the end sections,

$$\sigma_u=\sigma\,.\frac{1-\frac{15}{4}\frac{h}{f}}{1+\frac{45}{16}\frac{h^2}{f^2}}\quad\text{and}\quad\sigma_l=\sigma\,.\frac{1+\frac{15}{4}\frac{h}{f}}{1+\frac{45}{16}\frac{h^2}{f^2}}$$

Here the lower flange stress will be a maximum with $\frac{h}{f}=0.387$, giving $\sigma_l=1.72\,\sigma$. With $\frac{h}{f}\gtreqless\frac{4}{15}$, the upper flange at the end section receives a tensile stress. For illustration, if $\frac{h}{f}=\frac{1}{3}$, the stresses at the crown will be

$$\sigma_u=1.24\,\sigma,\qquad\sigma_l=0.28\,\sigma;$$

and the stresses at the end section will be

$$\sigma_u = -0.19\,\sigma, \qquad \sigma_l = 1.7\,\sigma.$$

If the half-span is loaded with p per unit length, we obtain, for the parabolic arch of constant section, $H = \frac{1}{16}\frac{p\,l^2}{f'}$; and the moments at the end-points of the axis are $M_1 = -\left(\frac{11}{8} - \frac{f}{f'}\right)\frac{p\,l^2}{24}$ and $M_2 = \left(\frac{f}{f'} - \frac{5}{8}\right)\frac{p\,l^2}{24}$. The greatest positive moment occurs at $x = \dfrac{\frac{13}{8} - \frac{f}{f'}}{2 - \frac{f}{f'}} \cdot \frac{l}{2}$ and is given by:

$$M_{max} = \frac{\frac{3}{2}\left(\frac{13}{8} - \frac{f}{f'}\right)^2 - \left(\frac{11}{8} - \frac{f}{f'}\right)\left(2 - \frac{f}{f'}\right)}{2 - \frac{f}{f'}} \cdot \frac{p\,l^2}{24}.$$

Neglecting the axial compression, i. e., with $f' = f$, we obtain

$$M_1 = -\frac{1}{64}p\,l^2 = -M_2, \quad \text{and } M_{max} = \frac{9}{1024}p\,l^2.$$

3. Segmental Arch. For a uniform load p extending for a distance λ from the left end, the end reactions V, $\mathbf{H}$ and M are obtained from equs. (362), (363) and (364) by substituting $G = p\,d\,x = -p\,r\cos\phi_1\,d\,\phi_1$ and integrating between the limits 0 and λ (or ϕ_0 and ϕ_1).

There will be given here only the formulae for the special case of the load p covering the entire span.

By symmetry, we obtain directly

$$V_1 = V_2 = \frac{p\,l}{2} = p\,r\sin\phi_0 \ldots\ldots\ldots\ldots\ldots\ (384.$$

We also have, by equ. (363),

$$\mathbf{H}_{tot} = \frac{p\,r\sin\phi_0}{6}\left(\frac{3(1-\delta)(\phi_0 - \sin\phi_0\cos\phi_0) - 2(1+\delta)\phi_0\sin^2\phi_0}{\phi_0(1+\delta)(\phi_0 + \sin\phi_0\cos\phi_0) - 2\sin^2\phi_0}\right). \ (385.$$

With this value, we obtain from equ. (364):

$$M_1 = \frac{p\,r^2}{12} \cdot \frac{3(1-\delta)(\phi_0 - \sin\phi_0\cos\phi_0)^2 - 2\sin^2\phi_0\left[(1+\delta)(3\phi_0^2 + \phi_0\sin\phi_0\cos\phi_0) - 4\sin^2\phi_0\right]}{\phi_0(1+\delta)(\phi_0 + \sin\phi_0\cos\phi_0) - 2\sin^2\phi_0} = M_2 \ \ldots (386.$$

For the moment at the crown, we obtain, by substituting the above values in (355),

$$M_c = \frac{p\,r^2}{12} \cdot \frac{3(1-\delta)(\sin^2\phi_0 - \phi_0^2 - 2\sin^2\phi_0\cos\phi_0 + 2\phi_0\sin\phi_0) - 4(1+\delta)\phi_0\sin^3\phi_0 + \sin^4\phi(1+3\delta)}{\phi_0(1+\delta)\phi_0 + \sin\phi_0\cos\phi_0) - 2\sin^2\phi_0} \ \ldots (387.$$

The maximum moments and shears are determined by the general rules. In the accompanying graphs, these values are plotted for a segmental arch having a uniform core-depth; Fig. 71 gives the maximum moments and Fig. 72 the maximum shears. The same notation is used as in the corresponding diagrams for the two-hinged and three-hinged arches; and since arches of the same rise and depth of rib have been considered, there is thus provided an opportunity for comparison of the three classes of arches with respect to the resultant stresses at the different sections. (cf. Figs. 50, 51; 62, 63.)

Fig. 71.

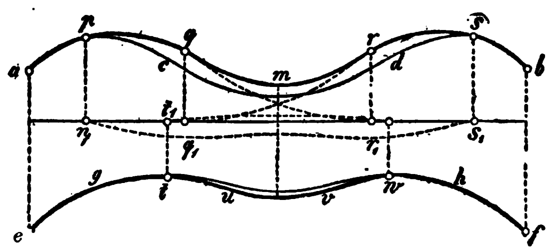
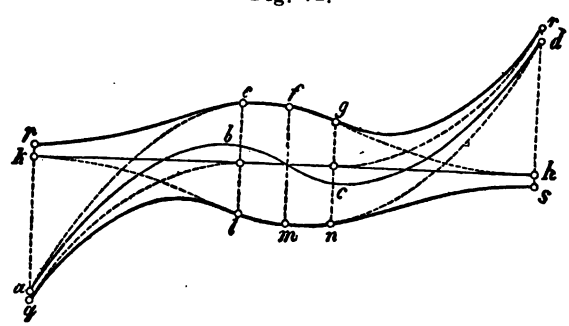


Fig. 72.



4. Temperature Stresses. The horizontal thrust produced by a variation of temperature has been derived in § 20:4; its value for the general case is given by equ. (337), for the parabolic arch by equ. (341), for the circular arch by equ. (363).

With regard to the magnitude of the temperature stresses, the following notes are given: In a parabolic arch-rib having two equal flanges of combined area A and effective depth h, the intensity of stress in the flanges at the crown is approximately

$$\sigma_c = \pm \frac{2 H_t \left(\frac{1}{3} f \mp \frac{h}{2}\right)}{A \, . \, h};$$

similarly, at the ends,

$$\sigma_1 = \mp \frac{2 H_t \left(\frac{2}{3} f \pm \frac{h}{2}\right)}{A \cdot h}.$$

Substituting H_t from equ. (341), we obtain

$$\sigma_c = \pm \frac{45}{24} E \omega t \frac{\left(1 \mp \frac{3}{2} \frac{h}{f}\right) \frac{h}{f}}{1 + \frac{45}{16} \frac{h^2}{f^2}}, \text{ and } \sigma_1 = \mp \frac{45}{12} E \omega t \frac{\left(1 \pm \frac{3}{4} \frac{h}{f}\right) \frac{h}{f}}{1 + \frac{45}{16} \frac{h^2}{f^2}}$$

For steel, we may use $E \omega t = 10,780$ pounds per square inch, thus giving, in round numbers,

$$\sigma_c = \pm 20,000 \frac{h}{f} \frac{1 \mp \frac{3}{2} \frac{h}{f}}{1 + \frac{45}{16} \frac{h^2}{f^2}}, \text{ and } \sigma_1 = \mp 40,000 \frac{h}{f} \frac{1 \pm \frac{3}{4} \frac{h}{f}}{1 + \frac{45}{16} \frac{h^2}{f^2}}.$$

The upper flange stress at the crown attains its greatest value with a ratio of $\frac{h}{f} = \frac{4}{15}$; we then have $\sigma_{cu} = + 2,670$ lbs. / sq. in. and $\sigma_{cl} = - 6,220$ lbs. / sq. in.; similarly the lower flange stress at the end section attains its maximum value with $\frac{h}{f} = 0.39$, when we find $\sigma_{1u} = - 14,040$ lbs. / sq. in. and $\sigma_{1l} = + 7,710$ lbs. / sq. in.

For a ratio of $\frac{h}{f} = \frac{1}{3}$, we find

$$\begin{cases} \sigma_{cu} = + \ 2,540, & \sigma_{cl} = - 7,620 \text{ lbs. / sq. in.} \\ \sigma_{1u} = - 12,700, & \sigma_{1l} = + 7,620 \text{ lbs. / sq. in.} \end{cases}$$

It thus appears that the temperature stresses may become very considerable and that, in this respect, the hingeless arch is at a disadvantage in comparison with the arch having hinged ends.

§ 22. Deformations.

The elastic deflections of any point $(x_1 y_1)$ of the arch, referred to the end A as origin of coordinates, are determined by equs. (185) and (186). Again employing the abbreviation

$$M\frac{I_0}{I'}+\left(\frac{P}{A}-E\,\omega\,t\right)\frac{I_0}{r\cos\phi}=z \quad \ldots\ldots\ldots\ldots (388.$$

[cf. equ. (293)], and conceiving these quantities z as forces acting on the arch, we obtain, exactly as for the two-hinged arch in § 19, the following expressions for the coordinate deflections:

$$\left.\begin{aligned}-E\,I_0\,\Delta\,y&=M_{zv}+C_1\\ E\,I_0\,\Delta\,x&=M_{zh}+C_2\end{aligned}\right\}\ldots\ldots\ldots\ldots (389.$$

Here M_{zv} and M_{zh} denote the static moments about the point $(x_1\ y_1)$ of the forces z applied at the points of the arch-axis together with a force $-Z_a=-E\,I_0\,\Delta\,\phi_0$ applied at the end A, these forces being applied vertically and horizontally for the respective moments. If the ends of the arch are perfectly immovable, then $\Delta\,\phi=0$, and hence $Z_a=0$. In general, Z_a is the end reaction produced by loading the horizontal projection of the arch with the forces z.

Furthermore, we may write

$$\left\{\begin{aligned}C_1&=I_0\int_{y_0}^{y_1}\left(\frac{P}{A}-E\,\omega\,t\right)dy\\ C_2&=-I_0\int_0^{x_1}\left(\frac{P}{A}-E\,\omega\,t\right)dx\end{aligned}\right.$$

or, with sufficient approximation,

$$\left.\begin{aligned}C_1&=I_0\left(\frac{H}{A_0}-E\,\omega\,t\right)y_1\\ C_2&=-I_0\left(\frac{H}{A_0}-E\,\omega\,t\right)x_1\end{aligned}\right\}\ldots\ldots\ldots\ldots (390.$$

Hence exactly the same rules apply to the determination of deflections in the hingeless arch as in the two-hinged type; and the contribution of the bending moments alone may be

obtained, exactly as in the preceding case, as the ordinates of a funicular polygon constructed with $E\,I_0$ as its pole distance for a loading consisting of the reduced moment-diagram of $M\frac{I_0}{I'}$. For the graphic determination, proceed as follows:

For the given loading, construct the equilibrium polygon or pressure line. The vertical intercepts m between this polygon and the arch-curve, multiplied by the ratio $\frac{I_0}{I'}$ and increased by the small and nearly constant correction

$$c = \left(1 - \frac{E\,\omega\,t\,A_0}{H}\right)\frac{I_0}{A_0 r \cos\phi} \dots\dots\dots\dots\dots \text{(391.}$$

will give the load-quantities

$$\frac{z}{H} = \left(m\frac{I_0}{I'} + c\right) \dots\dots\dots\dots\dots\dots\dots\dots \text{(392.}$$

These have to be laid off in a force polygon, with proper observance of the algebraic signs (m is positive if the pressure line lies above the arch-axis), and the resulting funicular polygons for vertical and horizontal action of the forces constructed. If the pole distance is selected $= \frac{1}{n} \cdot \frac{E\,I^0}{H\,.\,a}$, a being the assumed uniform horizontal spacing of the z-loads, then the ordinates of the two funicular polygons, measured by n times the scale of lengths, will give the vertical and horizontal deflections, respectively, of the points of the arch under the given loading. To be precise, there should be added the corrections $\frac{C_1}{E\,I_0}$ and $\frac{C_2}{E\,I_0}$; but, as a rule, these quantities are too small to warrant consideration.

We may also represent the influence lines for the horizontal and vertical deflections of any arch-point $(x_1\,y_1)$ by constructing the curves of vertical deflections producible by a unit load acting vertically and horizontally, respectively, at the point $(x_1\,y_1)$. Compare § 19.

In the above formulae, the effect of temperature is included. If it is desired to treat this effect separately, an expression may be established for it which is identical with that for a two-hinged arch, equ. (305). This gives the upward deflection of any arch-point due to temperature variation by the formula:

$$E\,I_0\Delta y = H_t m_x + 2\left(E\,\omega\,t - \frac{H_t}{A_0}\right) I_0 y_1 \dots\dots \text{(393.}$$

Here H_t denotes the horizontal thrust due to the change of temperature and m_x the static moment, about a vertical line through the given point, of the area included between the arch-curve and its rectifying line or line of action of the horizontal

thrust. This quantity is proportional to the ordinate (η) of the H-curve (Fig. 65^e); to be specific, $m_x = \eta \, . \, p \, . \, a$, where p is the pole distance and a the panel-length used in constructing the H-curve, and η is measured to the scale of lengths. We thus obtain for the temperature effect, in general,

$$\Delta y = (a \, . \, \eta + \beta \, . \, y) \, t \ldots\ldots\ldots\ldots\ldots\ldots\ldots (394.$$

For the example executed in Fig. 65, in which $l = 160$ m., $f = 42.5$ m., $A_0 = 0.112$ sq. m. and $I_0 = 0.70$ m.^{4}, we find by the approximate formula (341), assuming $E \omega = 248$ tonnes per sq. m. per degree C. and $\Delta l = 0$, $H_t = 1.040 \times t$ tonnes. The H-curve was constructed with $a = 16$ m. and $p = 25$ m., so that $H_t \, m_x = 16 \times 25 \times 1.04 \; \eta \, . \, t$ and $E \, I_0 \, \Delta \, y = (416 \eta + 334.2 \, y) \, t$; or, finally,

$$\Delta y \text{ (in mm.)} = (0.0297 \, \eta + 0.0238 \, y) \; t.$$

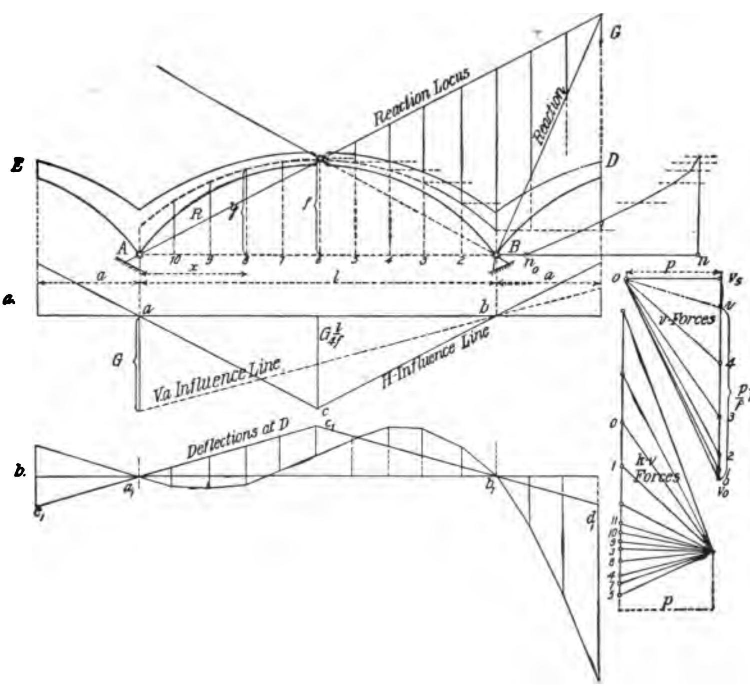
4. THE CANTILEVER ARCH.

§ 23. Arch With Hinged Ends, Having Cantilever Extensions Beyond the Points of Support.

The general equations (200) and (250) for determining the horizontal thrust hold good for loads on either cantilever arm if we substitute for **M** in those formulae the moments which would be produced in the span l of a simple beam.

In the *three-hinged arch* (Fig. 73), $H = \frac{\mathbf{M}_0}{f}$ and is pro-

Fig. 73.



portional to the moment $\mathbf{M}_0$ at the crown-hinge; hence the influence line for H is formed by prolonging the sides of the triangle, $c\,a$ and $c\,b$. The reaction lines are given by the rectilinear prolongations of the reaction locus. For maximum moments, only one of the cantilever arms has to be fully loaded,

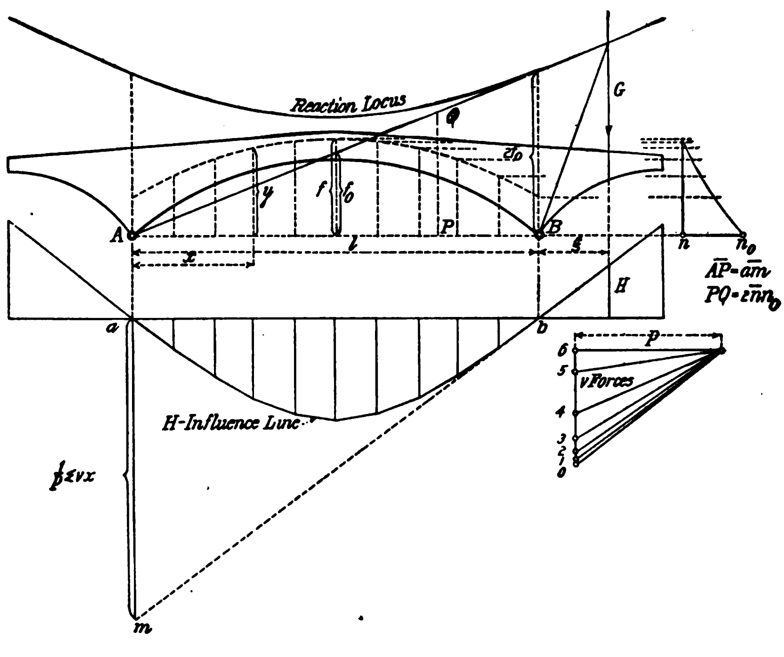
namely that one adjacent to the unloaded segment of the interior span. If the right arm is completely loaded with p per unit length, then, with the notation of Fig. 73, we obtain:

$$\left.\begin{aligned} H &= -\frac{1}{4}\frac{pa^2}{f} \\ M &= \frac{1}{4}\frac{pa^2}{f}\left(\frac{2fx}{l} - y\right) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (395.$$

If both cantilever arms are fully loaded,

$$\left.\begin{aligned} H &= -\frac{1}{2}\frac{pa^2}{f} \\ M &= -\frac{1}{2}\frac{pa^2}{f}(f-y) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (396.$$

Fig. 74.



In the *two-hinged arch,* the horizontal thrust may be determined by equ. (250); introducing the abbreviations N for the denominator and $\frac{y}{I} = v$, and retaining only the terms depending on the external loading, equ. (250) may be written

$$H = \frac{\sum\limits_0^l \mathbf{M} v}{N} \cdot$$

For a load G placed on the cantilever arm, $\mathbf{M} = -G\,\frac{\xi x}{l}$; hence

$$H = -G\frac{\xi}{l}\frac{\sum\limits_0^l v x}{N} \quad \text{.......................} (397.$$

In the H-influence line, Fig. 54, obtained as the funicular polygon of the v-forces according to § 17, the intercept of the prolongation of the last side upon the vertical through A gives the quantity $\sum\limits_0^l v\,x$; hence, by equ. (397), this line also constitutes the continuation of the H-influence line on the side of the cantilever arm (Fig. 74).

If the arch possesses a vertical axis of symmetry, then $\sum\limits_0^l v\,x = \frac{l}{2}\sum\limits_0^l v$, hence

$$H = -\frac{1}{2}G\,\xi\frac{\sum\limits_0^l v}{N} \quad \text{......................} (397^a.$$

The vertical component of the reaction at A is

$$V_1 = -G\frac{\xi}{l},$$

hence the direction of the left reaction is determined by

$$\tan\tau = \frac{V_1}{H} = \frac{N}{\sum\limits_0^l v x}$$

and is therefore independent of the position of the load G. Consequently the reaction locus is continued over the cantilever arm as a straight line consisting of the end-tangent from the point A.

If we substitute $2\,f_0 = l\tan\tau$, then

$$f_0 = \frac{l}{2}\frac{N}{\sum\limits_0^l v x} = \frac{N}{\sum\limits_0^l v}; \quad \text{...................} (398.$$

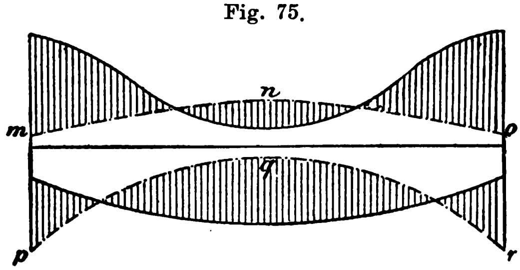
consequently,

$$H = -\frac{\xi}{2 f_0}G \quad \text{..........................} (399.$$

Thus, for loading on the cantilever arms, the two-hinged arch acts exactly like a three-hinged arch having its crown-hinge at a height of f_0; H and M may also be calculated by the same formulae (395) and (396), provided f_0 be substituted for f.

If a load p completely covers both cantilever arms, assumed equal in length, the moments at the core-points of the arch-sections are easily derived. They are, in fact, proportional to the distances of the core-lines from a horizontal line drawn at a height of f_0 above the end-points, and are given by these intercepts multiplied by $H = -\frac{1}{2}\frac{p\,a^2}{f_0}$ [cf. equ. (396)]. In Fig. 75, these moments are represented in the graphs $m\,n\,o$

Fig. 75.

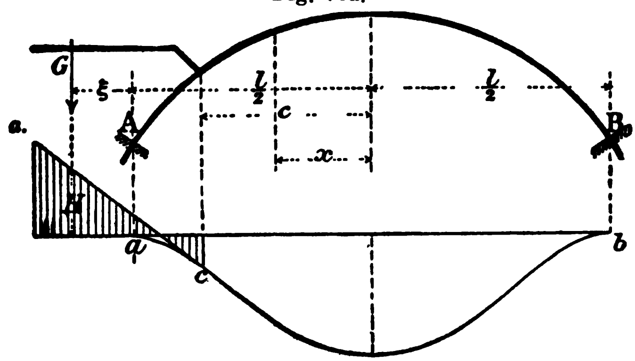


and $p\,q\,r$. Adding to these the moments due to a load p' covering the entire span l of the arch, we obtain the combined moments for the total loading as indicated by the shaded ordinates of Fig. 75; there is thus shown graphically the effect of the cantilever arms upon the distribution of stresses in the arch.

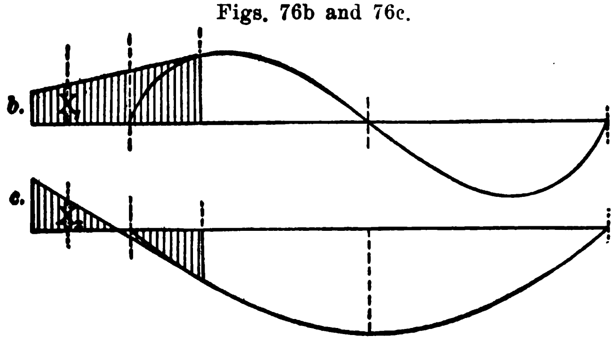
§ 24. General Case of the Cantilever Arch.

Let an arch with fixed ends have a cantilever arm anchored to it at any point (Fig. 76). The general equations (311) to

Fig. 76a.



Figs. 76b and 76c.



(313) apply here also, and may be written in an abbreviated form as follows:

$$\begin{cases} H = \dfrac{\sum\limits_0^l \mathbf{M}\, v}{N} \\[2ex] X_1 = \dfrac{\sum\limits_0^l \mathbf{M}\, v'}{\sum\limits_0^l x\, v'} \\[2ex] X_2 = \dfrac{\sum\limits_0^l \mathbf{M}\, v''}{\sum\limits_0^l v''} \end{cases}$$

where **M**, as usually, represents the moments within the span l of a simply supported beam. For the concentration G,

$$\mathbf{M} = G\,\frac{\xi + l}{l}\left(\frac{l}{2} - x\right)\Bigg]_{x=c}^{x=\frac{l}{2}}$$

$$\mathbf{M} = -\,G \cdot \frac{\xi}{l}\left(\frac{l}{2} + x\right)\Bigg]_{x=-\frac{l}{2}}^{x=c}$$

consequently

$$\sum_{0}^{l} \mathbf{M}\, v = G\left[\frac{\xi + l}{l}\sum_{c}^{\frac{l}{2}} v\left(\frac{l}{2} - x\right) - \frac{\xi}{l}\sum_{-\frac{l}{2}}^{c} v\left(\frac{l}{2} + x\right)\right]$$

The summations appearing in the above expression are represented as static moments of the v-forces by the intercepts of the corresponding sides of the funicular polygon upon the verticals through the ends A and B; i. e., the two summations are represented respectively by the end ordinates of a line drawn tangent to the H-curve at the point c (Fig. 76[a]). Hence, according to the equation

$$H = \frac{1}{N}\left[\frac{\xi + l}{l}\sum_{c}^{\frac{l}{2}} v\left(\frac{l}{2} - x\right) - \frac{\xi}{l}\sum_{-\frac{l}{2}}^{c} v\left(\frac{l}{2} + x\right)\right].G, \ldots (400.$$

the ordinate of the above tangent under the load G will give the horizontal thrust H measured to a scale of $G = N$.

The same relations hold good for the quantities X_1 and X_2 and their influence lines constructed as funicular polygons of the forces v' and v''. Here, again, the tangents to the respective influence lines at the springing-point of the cantilever arm give, by their ordinates, the values of X_1 and X_2 for any loading.

If $c = \frac{l}{2}$, i. e., with the cantilever arm attached at the end of the span, we find $H = 0$, $X_1 = -\,G\frac{\xi}{l}$, $X_2 = -\,G\frac{\xi}{2}$; hence $V_1 = G$, $V_2 = 0$, and $M_1 = M_2 = 0$, and we thus find, what should be self-evident, that the loading on such cantilever arm produces no effect on the arch with fixed ends.

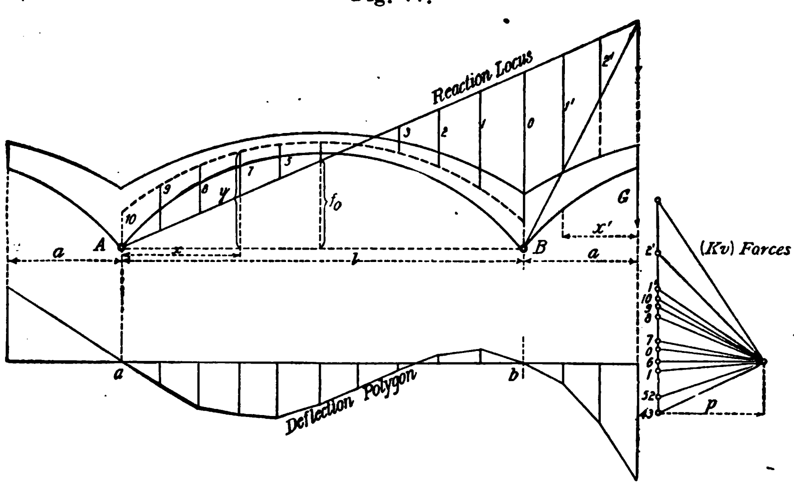
If the cantilever arm has its springing at the crown, we find H is constant for all values of ξ, i. e., independent of the position of the load. This applies also to the two-hinged arch.

§ 25. Deformations.

There will be considered here, in detail, only the determination of the vertical deflections of the end of the cantilever arm in the two-hinged arch represented in Fig. 77. For this purpose we may refer to the discussions given in § 19 and apply directly the graphic treatment there deduced.

Neglecting small correction terms, the deflections of the points of the arch under any loading are obtained as the static moments of a beam loaded with the forces $z = \frac{M}{EI}$; the resulting curve of deflections for a unit load acting at the end of the

Fig. 77.



cantilever arm will be, at the same time, by Maxwell's principle of reciprocal deflections, the influence line for the deflections of the free end produced by a concentration moving along the arch-rib.

By § 19, this influence line may be represented by the ordinate-differences of two funicular curves of which one is the H-influence line and the other is constructed for the load-quantities

$$k = \frac{1}{H_a} \frac{M}{I} = 2f_0 \cdot \frac{x}{l} \cdot \frac{1}{I}\Big]_{x=0}^{x=l} \quad \text{or} \quad k = 2f_0 \cdot \frac{x'}{a} \cdot \frac{1}{I}\Big]_{x'=a}^{x'=0}$$

We may also, as in Fig. 77, employ the differences of the load-ordinates

$$(k-v)=\frac{1}{I}\left(\frac{2f_0x}{l}-y\right)_{x=0}^{x=l} \qquad \text{or} \quad (k-v)=\frac{1}{I}\cdot\frac{2f_0x'}{a}\Big]_{x'=a}^{x'=0}$$

in constructing the deflection curve directly.

If, instead of the moment of inertia I, we substitute $\frac{I}{I_0}$ (where I_0 is any assumed constant moment of inertia), then the scale unit for the ordinates of the deflection curve will be $\frac{E\,I_0}{H_a.\,p.\,\Delta x}$ times the unit of the scale of lengths. Here $H_a=\frac{a}{2\,f_0}$ denotes the horizontal thrust for a unit load G at the end of the cantilever arm, $\Delta\, x$ is the horizontal spacing of the k and v ordinates, and p is the pole distance used in constructing the funicular polygon.

The expression for the deflections $(-\Delta\, y)$ of the end of the cantilever arm, due to a load G at any point x, as given by equ. (300), may be written as follows if we neglect the term depending on the axial thrusts:

$$-E\,I.\Delta y=G\,\mathrm{m}_x-H_a.\,m_x.$$

With a constant moment of inertia, we obtain:

$$\left.\begin{aligned}E\,I.\Delta y&=\left[\frac{1}{3}\,x(l^2-a^2-x^2)\,\frac{f_0}{l}-m_x\right]G\,\frac{a}{2f_0}\ \ (\text{for } x\text{=0 to } l)\\ E\,I.\Delta y&=-\left[\frac{1}{3}\,x'^3\,\frac{f_0}{a}-\frac{N}{2f_0}.\,(a-x')\right]G\,\frac{a}{2f_0}\ (\text{for } x'\text{=0 to } a)\end{aligned}\right\}..(401.$$

m_x denotes the moment of the quantities v; i. e., if u is the ordinate of the H-curve drawn with a pole distance, p, $m_x=u\,.\,p\,.\,\Delta\,x$. Again, N is the denominator of the expression for H.

The vertical displacement of the end of the cantilever due to a change of temperature, if we neglect the expansion of the cantilever itself, is given by

$$E\,I.\,\Delta\,y_t=-H_t\,\frac{N}{2\,f_0}\,.\,a,$$

or, substituting the value of H_t from equ. (290),

$$\Delta\,y_t=-\omega\,t\,b\,.\,cos\,\beta\,.\,\frac{a}{2\,f_0}\,\ldots\ldots\ldots\ldots\ldots\ldots(402.$$

If the arch has a *crown-hinge,* we may first neglect the interruption of continuity at the crown and, taking $k-v=0$ at this point, proceed as above to obtain a curve of deflections as the funicular polygon of the forces $k-v$. To the deflections thus determined we must add the effect of the rotation at the crown. This produces a crown sag, $-\,\delta$; and, by equ. (143), neglecting the term referring to the axial thrusts and with the altered

notation $\mu = \Sigma\, vy = N$, $\xi = a$, $m_\xi = \frac{a}{2f_0} . N$, we obtain

$$E\, I_0 \delta = G . \frac{a}{2f}\left(1 - \frac{f}{f_0}\right)\frac{l}{4f} . N \ldots\ldots\ldots\ldots\ldots (403.$$

Using the same scale as for the ordinates of the deflection curve, namely $\frac{E\, I_0}{H_0 . p . \Delta x} = \frac{E\, I}{p . \Delta x} . \frac{2f}{a}$ times the scale of lengths, the deflection ordinates due to the crown-hinge rotation are given by the moment ordinates for a crown-load of $\left(1 - \frac{f}{f_0}\right)\frac{N}{f . \Delta x}$ which, since $N = p . \Delta\, x . 2\, \overline{n_0\, n}$ and $f_0 = \frac{N}{\Sigma\, v}$ $= \frac{\overline{n_0 n} . p}{\overline{v_0 v_8}}$, is represented by a distance $2\left(\frac{p}{f}\,\overline{n\, n_0} - \overline{v_0\, v_8}\right)$ $= -2\,\overline{r\, v_8}$ (Fig. 73*b*). The deflection lines $d_1\, c_1\, e_1$ are therefore fixed by drawing $d_1\, c_1 \parallel Or$, and the resulting (full-drawn) intercepts of Fig. 73*b* give, by the above scale, the deflections of the cantilever end D for a moving concentration. (Compare equ. (143) et seq.)

The effect of temperature, as before, is given by

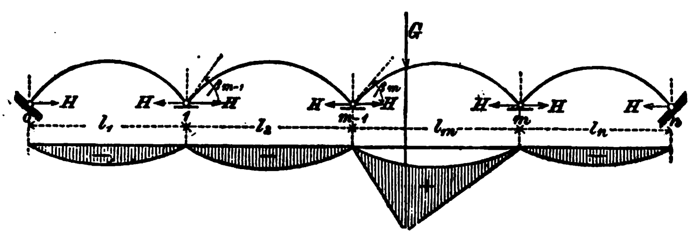
$$\Delta\, y_t = -\, \omega\, t\, b . \cos\beta . \frac{a}{2f} \ldots\ldots\ldots\ldots\ldots\ldots (402^a.$$

5. THE CONTINUOUS ARCH.

§ 26. Continuous Arch with Hinged Connections Between Successive Ribs. Determination of the Horizontal Thrust.

By continuous arches we understand a form of construction in which several arch-ribs have common intermediate piers on which their ends are free to slide together, but with the outer-

Fig. 78.



most ends definitely connected to the abutments. Let us first consider the case of such a bridge composed of n successive arch-ribs with hinged ends. Again allowing the approximation of writing $P' = H$ and of substituting a mean cross-section A_1, A_2, ..., for each rib, also writing for the mth span $\int \frac{1}{A}\left(\frac{y}{r}\, d\, s - d\, x\right) = -\frac{b_m \,.\, \cos\beta_m}{A_m}$ and $\int \frac{y^2\, d\, s}{I} = \sum_m \frac{y^2\, d\, s}{I}$, and, finally, assuming that the displacements on the piers may occur without friction and that the changes of temperature from the unstressed condition amount to t_1, t_2, t_3.. for the respective arches, we obtain for any loading on the mth span the following expression for the increase in length of that span:

$$\Delta\, l = \Sigma \frac{\mathrm{M}\, y}{E\, I} . d\, s - H \sum_m \frac{y^2\, d\, s}{E\, I} - H \,.\, \frac{b_m \cos\beta_m}{E\, A_m} + \omega\, t_m\, b_m \cos\beta_m .$$

Adding together all such expressions for the individual spans, we obtain:

$$\Sigma \Delta l = 0 = \sum_m \frac{My}{EI} ds$$
$$- H\left[\sum_1 \frac{y^2 ds}{EI} + \sum_2 \frac{y^2 ds}{EI} + \dots + \sum_n \frac{y^2 ds}{EI}\right]$$
$$- H\left[\frac{b_1 \cos \beta_1}{E A_1} + \frac{b_2 \cos \beta_2}{E A_2} + \dots + \frac{b_n \cos \beta_n}{E A_n}\right]$$
$$+ \omega\left[t_1 b_1 \cos \beta_1 + t_2 b_2 \cos \beta_2 + \dots + t_n b_n \cos \beta_n\right]$$

Hence,

$$H = \frac{\Sigma \frac{My}{I} \cdot ds + E\omega\,(t_1 b_1 \cos \beta_1 + t_2 b_2 \cos \beta_2 + \dots + t_n b_n \cos \beta_n)}{\sum_1 \frac{y^2 ds}{I} + \sum_2 \frac{y^2 ds}{I} + \dots + \sum_n \frac{y^2 ds}{I} + \frac{b_1 \cos \beta_1}{A_1} + \frac{b_2 \cos \beta_2}{A_2} + \dots + \frac{b_n \cos \beta_n}{A_n}} \quad (404.$$

Comparing this with the case of the simple two-hinged arch we find the following: The horizontal thrust due to any load is essentially less in the continuous arch; this reduction of thrust increases with the number of arches linked together. If all the arches have the same dimensions, then for n arches for which only one is loaded,

$$H = \frac{1}{n} H' \quad \dots\dots\dots\dots\dots\dots\dots\dots \quad (404^a.$$

where H' denotes the thrust in the single loaded arch if its ends were immovable.

For constructing the H-influence line, the procedure given in § 17 may be directly applied. In obtaining the maximum stresses, the method of influence lines recommends itself; and it is thus made apparent that the maximum positive moment at any section will occur when the corresponding span alone is partially or entirely loaded, whereas, for the greatest negative moment, all the remaining spans must be completely loaded.

If the frictional resistance at the piers is considered, the calculation of the horizontal thrust is changed as follows: Let H_1 be the horizontal thrust transmitted to the left abutment; also let it be assumed that the vertical pressures on the intermediate piers due to the dead weight of the arch-ribs are practically uniform and equal to A, and that the coefficient of friction $= f$. Let the loading in the mth span produce on the $(m-1)$th and mth piers the vertical reactions V_1 and V_2 respectively. Then the horizontal forces acting in the first to the nth spans are:

$$H_1, H_1 + fA, H_1 + 2fA \dots, H_1 + (m-2) fA, H_1 + (m-1) fA + fV_1,$$
$$H_1 + (m-2) fA + f(V_1 - V_2), H_1 + (m-3) fA + f(V_1 - V_2) \dots,$$
$$H_1 + (2m - n - 1) fA + f(V_1 - V_2).$$

Hence, retaining the same simplifications as above and omit-

ting the effect of temperature as given by equ. (404), we obtain, from the equation of work, the following expression for H_1:

$$H_1 = \frac{\Sigma \frac{Myds}{I} - C_1 . f . A - C_2 f V_1 - C_3 f (V_1 - V_2)}{\underset{1}{\Sigma} \frac{y^2 ds}{I} + \underset{2}{\Sigma} \frac{y^2 ds}{I} + \ldots \underset{n}{\Sigma} \frac{y^2 ds}{I} + \frac{b_1 \cos \beta_1}{A_1} + \frac{b_2 \cos \beta_2}{A_2} + \ldots \frac{b_n \cos \beta_n}{A_n}} \quad \ldots (405.$$

Here,

$$\left.\begin{aligned} C_1 = \Big[& \underset{2}{\Sigma} \frac{y^2 ds}{I} + 2 \underset{3}{\Sigma} \frac{y^2 ds}{I} + \ldots (m-1) \underset{m}{\Sigma} \frac{y^2 ds}{I} \\ & + (m-2) \underset{m+1}{\Sigma} \frac{y^2 ds}{I} + \ldots (2m-n-1) \underset{n}{\Sigma} \frac{y^2 ds}{I} + \frac{b_2 \cos \beta_2}{A_2} \\ & + 2 \frac{b_3 \cos \beta_3}{A_3} \ldots + (m-1) \frac{b_m \cos \beta_m}{A_m} + (m-2) \frac{b_{m+1} \cos \beta_{m+1}}{A_{m+1}} \\ & + \ldots (2m-n-1) \frac{b_n \cos \beta_n}{A_n} \Big] \\ C_2 = & \underset{m}{\Sigma} \frac{y^2 ds}{I} + \frac{b_m \cos \beta_m}{A_m} \\ C_3 = & \underset{m+1}{\Sigma} \frac{y^2 ds}{I} + \ldots \underset{n}{\Sigma} \frac{y^2 ds}{I} + \frac{b_{m+1} \cos \beta_{m+1}}{A_{m+1}} + \ldots \frac{b_n \cos \beta_n}{A_n} \end{aligned}\right\} \ldots (406$$

If, as before, we assume all the arches to have the same cross-section, then

$$\left.\begin{aligned} C_1 &= \frac{1}{2} \left[2m(2n-m+1) - n^2 - 3n\right] \left(\Sigma \frac{y^2 ds}{I} + \frac{b \cos \beta}{A}\right) \\ C_2 &= \Sigma \frac{y^2 ds}{I} + \frac{b \cos \beta}{A} \\ C_3 &= (n-m) \left(\Sigma \frac{y^2 ds}{I} + \frac{b \cos \beta}{A}\right) \end{aligned}\right\} \ldots (406^a.$$

and we obtain

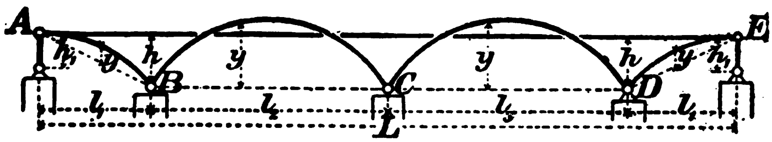
$$\left.\begin{aligned} H_1 = \frac{1}{n} H' - \frac{2m(2n-m+1) - n^2 - 3n}{2n} f . A - \frac{1}{n} f V_1 \\ - \frac{n-m}{n} f (V_1 - V_2) \end{aligned}\right\} \ldots (405^a.$$

It should be observed, however, that the horizontal thrust due to loading can never be negative in any span, i. e., the horizontal thrust, already existing on account of dead load cannot become diminished. If the contrary is indicated by the above formula, then we must consider as included in the system only those arches on each side of the loaded span for which the formula gives a positive value of the horizontal thrust; H_1 must

be positive in the outermost spans. The horizontal force in the loaded span is

$$H = H_1 + (m-1) f . A + f . V_1 .$$

Fig. 79.



A special form of the continuous arch, designated by *Haberkalt** as the "Balanced Arch," is shown in Fig. 79. The end spans consist of half-arches whose free ends are connected together through a straight tension rod $A\,E$. This tie-rod has no connection with the intermediate arches. Only one of the supports is fixed, the others are free to slide horizontally; in the terminal supports, A and E, this freedom of movement may be attained through the use of rocker-piers. To calculate the force H acting in the tie-rod, we may again employ equ. (404), if we denote by y the arch-ordinates measured from the respective chords and add to the denominator a term $\frac{L}{A}$ representing the effect of the tensile strain in the tie-rod. (L = length and A = cross-section of the tie-rod $A\,E$.) If the ends A and E rest on rocker-piers of section A_1 and height h_1 and if l_1 and h are the horizontal and vertical distances between the end-points of each terminal arch, there enters into the denominator of equ. (404) an additional term: $\frac{2\,h_1\,h^2}{A_1\,l_1^2}$. For a load G located in either end span at a distance ξ from A or E, the numerator of the expression for H must be increased by the small term

$$G . \frac{(l-\xi)\,h . h_1}{A_1\,l_1^2} .$$

The effect of a temperature change of $t°$ is represented by adding to the numerator of equ. (404) the term

$$-E\,\omega\,t\left(L + \frac{2\,h\,h_1}{l_1}\right).$$

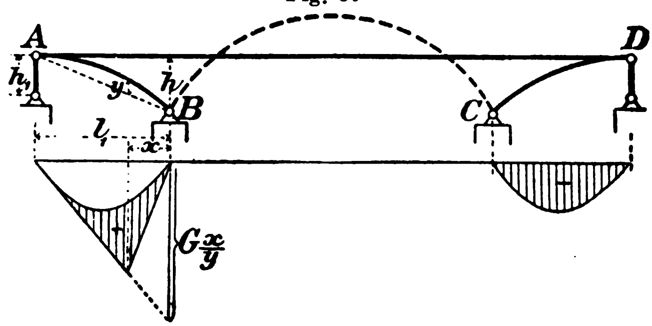
This system of arches has found application, in the form shown in Fig. 80, in a foot-bridge over the Seine built at Paris in 1900. In this case, however, the supports on the two intermediate piers are made fixed; so that the middle arch, acting as

* Österr. Wochenschr. f. d. öffentl. Baudienst, 1901.

an ordinary two-hinged arch, is excluded from the continuous structure. The tension in the tie-rod is determined by

$$H = \frac{1}{2}\,\frac{\Sigma\frac{M\,y}{I}\,ds + G(l-\xi)\,\frac{h\,h_1}{A_1\,l_1^2}}{\Sigma\frac{y^2}{I}\,ds + \frac{b\cos\beta}{A_0} + \frac{L}{2A} + \frac{h_1\,h^2}{A_1\,l_1^2}}$$

Fig. 80.



In the above expression, the summations cover a single side-span and the quantities A, A_1, h_1, etc., have the denotation previously assigned.

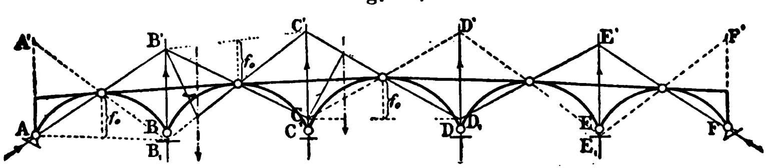
In Fig. 80 the shaded ordinates give the influence of a concentration G upon the moment at the arch-point M measured to a scale whose unit is $\frac{G}{y}$.

§ 27. The Continuous Arch with Hingeless Connections between the Ribs.

If the successive arches are rigidly connected together at the intermediate piers, so that any change in the angle between the arch-axes at the points of support is prevented, although a horizontal displacement at these points is permitted, the structure thus formed of n successive spans without any hinges is of n-fold static indeterminateness. It is here assumed that the connection to the abutments possesses freedom of rotation.

1. A **statically determinate** structure is obtained if a hinge is inserted in each span. However, in general, such a system will be practicable only with an *odd* number of spans; for, if the number of spans is even, the system lacks stability. The

Fig. 81.



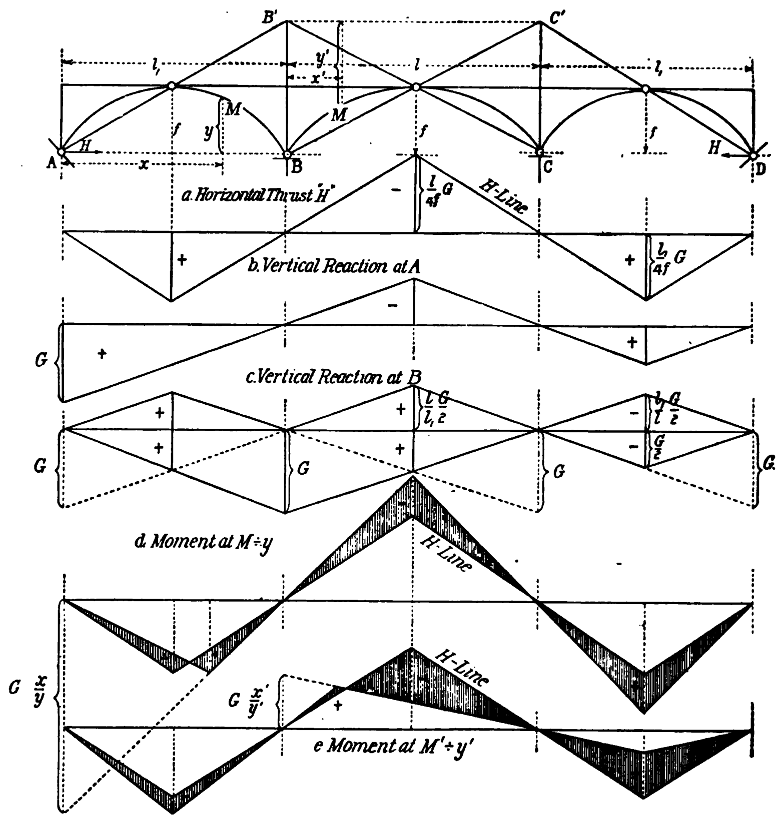
system ceases to be a rigid one, but enters into unstable equilibrium, when it is possible to connect the end-hinges by a chain of straight lines passing through all the intermediate hinges (the chain of lines $A\,B'\,C_1\,D'\,E_1\,F'$ in Fig. 81). If the last point (F') does not coincide with the end-support (F), the stability of the system is assured; but the capability of deformation is even then considerable, if F' approaches close to the point F.

If only one span, as $B\,C$ in Fig. 81, is loaded, the reactions in the unloaded spans are given by the chains of lines from the abutments, $A\,B'$ and $F\,E'\,D_1\,C'$, while the loaded span acts as a three-hinged arch supported at B' and C'. In the second, fourth, and all the even spans, these imaginary points of support lie *above* the crown-hinge; in the odd-numbered spans these points fall *below* the crown-hinge; and, in such case, they will coincide with the actual pier-supports B, C, D.., if all the crown-hinges are located at the same height above the closing chords $A\,B\,C\,D$.., i. e., if all the arches have the same rise. Loading in the odd spans will thus produce *compression* at the end abutments, and loading in the even spans will produce

tension. The individual spans act alternately as erect or inverted arches of equal virtual rise f_0 throughout. All parts of the structure are thus subjected to reversals of stress.

For a moving concentration, the reactions vary as a linear function of the position of the load; and, for a structure of three spans of equal rise, the influence lines for the horizontal thrust

Fig. 82.



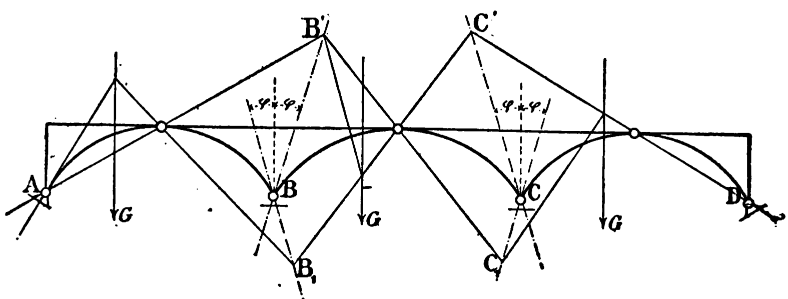
at the end-supports and for the vertical reactions at A and B are shown in Fig. 82, *a, b, c*.

The moment, referred to the point (x, y) in the first span or to the point (x', y') in the second span, is given by $M = \mathbf{M} - Hy$ or $M = \mathbf{M} + Hy'$, respectively, where $\mathbf{M}$ denotes the moment of the vertical forces for a single arch. Hence the influence quantity for M is obtained in Fig. 82, *d* and *e*, as the difference between the ordinates of the H-curve and those of

the $\frac{M}{y}$ or $\frac{M}{y'}$ lines. These intercepts must be multiplied by the factors y or y'; and these ordinates y must be measured, in the odd spans, above the true points of support $A\,B$, and, in the even spans, from the virtual points of support $B'\,C'$. There thus remains no difficulty in establishing the analytical expressions for the resultant forces and moments.

If the friction at the movable intermediate supports is considered, then the reaction at those points must no longer be taken as vertical but as inclined at the angle of friction. The reactions of the loaded span tend to be displaced outward, but the direction of displacement changes alternately for the other spans. Since the reactions in these spans also alternate regularly between tension and compression, it appears that the directions of the reactions at all the piers on each side of the loaded span must be inclined toward that span (Fig. 83). The intermediate

Fig. 83.



supports of the successive spans are replaced by the imaginary points $B_1\,C_1$.., $B'\,C'$..; and the changes in the reactions from those of the frictionless condition are easily obtainable.

2. **If no crown-hinges are provided,** the structure with n distinct spans is n-fold statically indeterminate; and, to determine the reactions, n equations of elasticity must be established.*

To establish these equations of condition, it is simplest to make use of the principle of least work. For flat arches, with the depth of section small in comparison with the radius of curvature, the work of deformation may be written

$$W = \int \frac{M^2}{2\,E\,I}\,d\,x + \frac{H^2\,l}{2\,E\,A} - \omega\,H \int t\,d\,x, \dots . \quad (407.$$

* The theory of this type of structure, which may be considered as a more general case of the continuous beam, was first presented by *H. Müller-Breslau* (Wochenbl. f. Arch. u. Ing., 1884). In the following treatment, this work has been utilized.

if we adopt the approximations of taking the axial-force $= H$ and assuming a mean cross-section A, and denote by I the moment of inertia multiplied by $\cos\phi = \frac{dx}{ds}$. The integration is to cover the entire length of the structure and l denotes this total length of span between end abutments.

If $V_1\, V_2 \ldots$ denote the vertical reactions at the 1st, 2nd, .., intermediate piers, and if $y_1\, y_2 \ldots$ are the moments at any point of a straight beam of span l loaded with a unit force successively at the different points corresponding to the intermediate supports; if, furthermore, **M** is the bending moment producible in this beam by the external loads acting on the continuous arch and if y is the arch-ordinate measured from the line joining the end-points $A\,B$; then we have

$$M = \mathbf{M} - H\,y - V_1 y_1 - V_2 y_2 - V_3 y_3 - \ldots\ldots \quad (408$$

If, as a general case, we assume that, under the loading, the end-points are displaced outward by an amount Δl and the intermediate supports are depressed by the small amounts $\delta_1\, \delta_2 \ldots$, then *Castigliano's Theorem*, concerning the derivatives of the work of deformation, yields the following relations:

$$\Delta l = -\frac{\partial W}{\partial H}, \quad \delta_1 = -\frac{\partial W}{\partial V_1}, \quad \delta_2 = -\frac{\partial W}{\partial V_2} \ldots$$

But, by equ. (408),

$$\frac{\partial M}{\partial H} = -y, \quad \frac{\partial M}{\partial V_1} = -y_1, \quad \frac{\partial M}{\partial V_2} = -y_2 \ldots$$

Hence, if I_0 denotes an arbitrary moment of inertia, equ. (407) yields the following equations of condition:

$$E\,I_0 \Delta l - \omega\, E\,I_0\, t\, l = \int M \frac{I_0}{I}\, y\, dx - H\, l \frac{I_0}{A}$$

$$E\,I_0 \delta_1 = \int M \frac{I_0}{I}\, y_1\, dx$$

$$E\,I_0 \delta_2 = \int M \frac{I_0}{I}\, y_2\, dx.$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

If we substitute here the value of M from equ. (408) and introduce the abbreviations

$$\left.\begin{aligned} c_0 &= \int y^2 \frac{I_0}{I}\, dx + l \frac{I_0}{A} \\ c_m &= \int y\, y_m \frac{I_0}{I}\, dx \\ c_{mr} &= \int y_m\, y_r \frac{I_0}{I}\, dx \end{aligned}\right\}, \quad \ldots\ldots\ldots\ldots \quad (409.$$

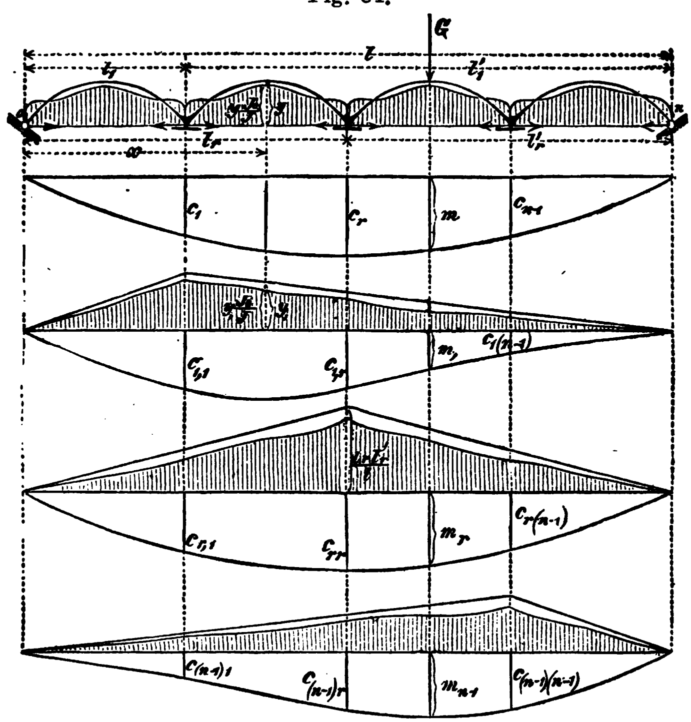
MM

we finally obtain the equations:

$$\left.\begin{array}{r}-\omega E I_0 t l + E I_0 \Delta l = \int \mathbf{M} \frac{I_0}{I} y\, dx - H c_0 - V_1 c_1 - V_2 c_2 - V_3 c_3 - \ldots \\ E I_0 \delta_1 = \int \mathbf{M} \frac{I_0}{I} y_1\, dx - H c_1 - V_1 c_{11} - V_2 c_{21} - V_3 c_{31} - \ldots \\ E I_0 \delta_2 = \int \mathbf{M} \frac{I_0}{I} y_2\, dx - H c_2 - V_1 c_{12} - V_2 c_{22} - V_3 c_{32} - \ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\end{array}\right\} \ldots (410.$$

If the load consists of a single concentration G in the mth span, the definite integrals $\int \mathbf{M} \frac{I_0}{I} y\, dx$, $\int \mathbf{M} \frac{I_0}{I} y_1 dx \ldots$,

Fig. 84.



may be represented by the ordinates of respective funicular polygons. They are in fact equal (cf. § 17) to G times the moments, about the load point, in a simple beam of span l loaded successively with the quantities $y \frac{I_0}{I}$, $y_1 \frac{I_0}{I}$, $y_2 \frac{I_0}{I} \ldots$(Fig. 84). Let us denote these moments, which may readily be determined by construction, by $m, m_1, m_2 \ldots$

The quantities $c_1\, c_2 \ldots c_m$ and the general term c_{mr} are also given by these funicular polygons, as follows: Since $y_r = \frac{l'_r}{l}\, x$ (from $x=0$ to $x=l_r$) and $y_r = \frac{l_r}{l}\,(l-x)$ (from $x=l_r$ to $x=l$), we have

$$c_{mr} = \frac{l'_r}{l}\int_0^{l_r}\left(y_m \frac{I_0}{I}\right) x\, dx + \frac{l_r}{l}\int_{l_r}^{l}\left(y_m \frac{I_0}{I}\right)(l-x)\, dx,$$

i. e. = the moment about the rth point of support of the area of the graph of $y_m \frac{I_0}{I}$; or, since by equ. (409) $c_{mr} = c_{rm}$, it is also equal to the moment about the mth point of support of the area of the graph of $y_r \frac{I_0}{I}$. The equations (410) thus take the form

$$\left.\begin{aligned} -\omega E I_0 t l + E I_0 \Delta l &= G m - H c_0 - V_1 c_1 - V_2 c_2 - V_3 c_3 - \ldots \\ E I_0 \delta_1 &= G m_1 - H c_1 - V_1 c_{11} - V_2 c_{21} - V_3 c_{31} - \ldots \\ E I_0 \delta_2 &= G m_2 - H c_2 - V_1 c_{12} - V_2 c_{22} - V_3 c_{32} - \ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\} \ldots \quad (411.$$

There remains to be determined the definite integral $c_0 = \int_0^l \left(y \frac{I_0}{I}\right) y\, dx + l \frac{I_0}{A}$. This quantity is nothing else than the moment, about the chord $A\,B$, of the load-quantities $y \frac{I_0}{I}\, dx$ applied horizontally at the respective points of the arch. Hence this quantity may, in a familiar manner, also be obtained by graphic construction; and the solution of equations (411) will yield the values of the reactions H, V_1, $V_2 \ldots$ If the effects of temperature and of yielding of the piers are, for the sake of simplicity in computation, omitted for separate consideration, then the left-hand members of the above equations must be replaced by 0, and the solution gives:

$$\left.\begin{aligned} H &= G\,(\alpha m + \beta m_1 + \gamma m_2 + \ldots) \\ V_1 &= G\,(\alpha_1 m + \beta_1 m_1 + \gamma_1 m_2 + \ldots) \\ V_2 &= G\,(\alpha_2 m + \beta_2 m_1 + \gamma_2 m_2 + \ldots) \\ &\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\} \ldots\ldots\ldots \quad (412.$$

where $\alpha\,\beta\,\gamma \ldots,\ \alpha_1\,\beta_1\,\gamma_1 \ldots,\ \alpha_2\,\beta_2\,\gamma_2 \ldots$ represent coefficients which are independent of the position of the load G; so that it

is necessary to solve the equations but once in order to obtain the influence of any position of the load.

Equations (411), with the omission of the first, also serve to determine the reactions for a continuous beam on putting $H = 0$.

For a more exact analysis, the friction at the supports should also be considered.

§ 28. More Exact Theory of the Arch, Including the Effect of the Deformations Under Loading.

As previously stated, in § 10, the foregoing theory of the arch-rib is founded on the approximation of assuming that the elastic curve under loading is identical with the initial form of the arch-axis; hence, in determining the moments, the deformation of the arch caused by the loading has not been considered. This is the same approximation as was made in the treatment of stiffened suspension bridges in §§ 4 to 8; and, for a more accurate analysis, we must follow the same general procedure developed in outline in § 9. However, since the error of the approximate theory diminishes as the deformations of the structure become less, it may be generally stated that in arches, which as a rule are more rigidly constructed than suspension bridges (especially the earlier examples), it will less often be necessary to apply the rigorous theory. We will therefore take up the exact analysis only so far as may be necessary in order to obtain an estimate of the admissibility of the approximate treatment and of the magnitude of its error.

Let us consider a two-hinged arch of uniform cross-section, and let (x, y) denote the coordinates of any point of the arch-axis, measured from the left point of support, in the unstressed condition of the arch-axis; and let the vertical deflection of the point, under loading, be η. We will neglect the horizontal deflections of the points of the arch and, furthermore, assume that the end-reactions are not affected by the deformations. With the usual notation, the moment at the given point of the arch may be written:

$$M = \mathbf{M} - H(y - \eta);$$

and, for a flat arch of approximately constant curvature, the change of curvature due to deflection will be, by equ. (183),

$$\left(1 - \frac{H}{EA}\right)\frac{d^2\eta}{dx^2} = -\frac{M}{EI},$$

or, substituting the above expression for M,

$$\left(EI - \frac{I}{A}H\right)\frac{d^2\eta}{dx^2} = -\left(\frac{\mathbf{M}}{H} - y\right)H - H\eta.$$

Substituting the abbreviations

$$c^2 = \frac{H}{EI - \frac{I}{A}H} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (413.$$

and $$F(x) = \frac{M}{H} - y,$$

the differential equation of the deflection-ordinates of the arch becomes

$$\frac{d^2\eta}{dx^2} + c^2\eta + c^2F(x) = 0.$$

For the cases of loading considered in practice and with a segmental or parabolic arch-axis, $F(x)$ is an algebraic function of no higher than the second degree, so that its second differential coefficient will be constant and its higher derivatives will vanish. Then the integral of the above differential equation is

$$\eta = A \sin cx + B \cos cx - F(x) + \frac{1}{c^2}F''(x) \quad .. (414.$$

with which the expression for the moment becomes

$$M = H\left(A \sin cx + B \cos cx + \frac{1}{c^2}F''(x)\right) \quad .. (415.$$

The constants A and B are determined by the condition that at $x = 0$ and $x = l$, $\eta = 0$; and, at any interruption of continuity, equal values must obtain for η and $\frac{d\eta}{dx}$.

To determine the horizontal thrust H, the following method may be used. Let us consider an arch loaded with a unit continuous load, and assume the resulting pressure-line to coincide with the axis of the arch; then, if we thus disregard the bending-moments, the axial thrust in the arch $= 1 . r$, where $r = \frac{l^2}{8f}$ is the radius of curvature of the arch at the crown. The actual distributed load q, in the place of the unit loading, produces the deflections η and, with the length of the axis $= b$, a shortening of the arch-axis by the amount $\frac{H}{EA} . b$. If we apply the principle of virtual work to the assumed load 1, on the basis of the deflections η, the equating of the external and the internal work gives $1.\int_0^l \eta\, dx = 1 . r . \frac{Hb}{EA}$.

Substituting in this equation the above expression for η (equ. 414) and observing that, for a parabolic arch uniformly loaded,

$$F''(x) = -\frac{q}{H} + \frac{8f}{l^2},$$

we obtain, with the aid of equ. (415),

$$H = \frac{\int_0^l \left(M + \frac{q}{c^2}\right) dx}{\int_0^l (A \sin cx + B \cos cx)\, dx + \frac{1}{c^2}\frac{8f}{l} + \frac{2}{3} fl - \frac{H}{EA} rb} \quad \ldots (416.$$

To apply this formula, an approximate value for H must first be calculated by the formulae previously established. [Compare this expression with that of equ. (152) for the case of the suspension bridge.]

For a uniformly distributed full-span load, the constants A and B are found, by the condition that $\eta = 0$ for $x = 0$ and $x = l$, to have the values

$$\begin{cases} A = \frac{1}{c^2}\left(\frac{q}{H} - \frac{8f}{l^2}\right) \tan \frac{1}{2} cl \\ B = \frac{1}{c^2}\left(\frac{q}{H} - \frac{8f}{l^2}\right). \end{cases}$$

Hence, by equ. (415),

$$M = \frac{1}{c^2}\left(q - H\frac{8f}{l^2}\right)\left[1 - \frac{\cos c\left(\frac{l}{2} - x\right)}{\cos \frac{1}{2} cl}\right] \quad \ldots (417.$$

and, by equ. (416),

$$H = \frac{\frac{1}{12} ql^2 + \frac{1}{c^2} ql}{\frac{2}{3} fl + \frac{1}{c^2}\left[\left(\frac{q}{H} - \frac{8f}{l^2}\right)\frac{2}{c} \tan \frac{cl}{2} + \frac{8f}{l}\right] - \frac{H}{EA} rb} \quad \ldots (418.$$

For very small values of c, hence for very rigid arches, we thus find, in agreement with the approximate theory,

$$M = \frac{1}{2} qx(l - x)\left(1 - H\frac{8f}{ql^2}\right)$$

and approximately (assuming $I = \infty$),

$$H = \frac{1}{8}\frac{ql^2}{f}.$$

At the limiting value of $\frac{1}{2} cl = \frac{\pi}{2}$, we find $M = \infty$, i. e. the arch must *fail under the bending-stress.* This yields the relation

$$c^2 l^2 = \pi^2 = \frac{Hl^2}{EI - \frac{I}{A} H}$$

or

$$H\left(1 + \frac{\pi^2 I}{Al^2}\right) = \frac{\pi^2 EI}{l^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (419.$$

The limiting value thus fixed for the horizontal thrust at which the arch will fail by bending is nearly identical with the

buckling load given by Euler's formula for straight columns whose unsupported length is l.

If the loading consists of a single concentration G, applied at distances ξ and ξ' from the respective ends of the span, the moment at any point to the left or right of the load may be written

$$M = H\,(A \sin c\,x + B \cos c\,x) + \frac{H}{c^2}\,\frac{8f}{l^2}$$

or

$$M' = H\,(A' \sin c\,x + B' \cos c\,x) + \frac{H}{c^2}\,\frac{8f}{l^2}\,.$$

The constants are determined by the conditions that, for $x = 0$ and $x = l$, $\eta = 0$, and, at the point of application of the load, η and $\frac{d\eta}{dx}$ must have the same values for the left and right segments of the rib. Substituting $\xi' = l - \xi$ and $x' = l - x$ we obtain

$$M = \frac{1}{c}\,\frac{\sin c\xi' \sin c x}{\sin c l}\,G - \frac{H}{c^2 r}\left(\frac{\sin c x + \sin c x'}{\sin c l} - 1\right) \quad \text{(420.}$$

which expression applies for all points from $x = 0$ to $x = \xi$. For the right segment of the arch, i. e., from $x = \xi$ to $x = l$, we must interchange ξ' with ξ and x' with x in the above expression.

For the horizontal thrust, equ. (416) yields the expression

$$H = \frac{\frac{1}{2}\,G\,.\,\xi\,.\,\xi'}{\frac{2}{3}\,f\,l + \frac{1}{c^2}\,\frac{G}{H}\,\frac{\sin c\xi + \sin c\xi' - \sin c l}{\sin c l} - \frac{1}{c^2}\,\frac{8f}{l^2}\left(\frac{2}{c}\tan\frac{c\,l}{2} - l\right) - \frac{H}{EA}\,r\,b} \quad \text{(421.}$$

With $c = 0$, i. e., for the infinitely rigid arch, the above expression gives $H = 0$. But if we assume c only so small that the arc-functions *sin* and *tan* may be replaced by the corresponding arcs, we obtain

$$H = \frac{\frac{1}{2}\,G\,\xi\,\xi'}{\frac{2}{3}\,f\,l - \frac{H}{EA}\,r\,b}\,;$$

and, if we neglect the relatively very small second term in the denominator, this formula becomes identical with that of *Engesser* and *Müller-Breslau*: $H = \frac{3}{4}\,G\,\frac{\xi\,\xi'}{f\,l}$ (see p. 128).

For this case, namely with $c = 0$, the terms of equ. (420) take the indeterminate form of $\frac{0}{0}$, whose evaluation by the usual

methods gives the following expression in agreement with the approximate theory:

$$M = G\frac{x\xi'}{l} - H.\frac{4f}{l^2}x(l-x).$$

If the arch is uniformly loaded over the entire span with g per unit length, and, in addition, with a load p extending for a distance λ from the left end, the following expressions for the moments at the arch-points are derived from equ. (415): From $x = 0$ to $x = \lambda$,

$$\left.\begin{aligned} M &= \frac{H}{c^2 \sin cl}\Big[\left(\frac{g}{H} - \frac{8f}{l^2}\right)(\sin cx + \sin cx' - \sin cl) \\ &\qquad + \frac{p}{H}(\sin cx' + \sin cx \cos c\lambda - \sin cl)\Big] \\ &\text{From } x = \lambda \text{ to } x = l, \\ M &= \frac{H}{c^2 \sin cl}\Big[\left(\frac{g}{H} - \frac{8f}{l^2}\right)(\sin cx + \sin cx' - \sin cl) \\ &\qquad + \frac{p}{H}(1 - \cos c\lambda)\sin cx'\Big] \end{aligned}\right\} \quad \text{.. (422.}$$

Here, $x' = l - x$ and $\lambda' = l - \lambda$.

For the horizontal thrusts in this case, equ. (416) yields the expression:

$$H = \frac{\frac{1}{12}\left[gl^3 + p\lambda^2(3l - 2\lambda)\right] + \frac{gl + p\lambda}{c^2}}{K + \frac{1}{c^2}\frac{8f}{l} + \frac{2}{3}fl - \frac{H}{EA}rb} \quad \text{......... (423.}$$

where

$$K = \frac{1}{c^3 \sin cl}\left[2\left(\frac{g}{H} - \frac{8f}{l^2}\right)(1 - \cos cl) + \frac{p}{H}(1 + \cos c\lambda' - \cos cl - \cos c\lambda)\right]$$

For comparison of these formulae with those of the approximate theory, the following example is worked out:

For a parabolic arch of constant section, let $l = 120$ m., $f = 12.13$ m., $A = 0.06696$ m^2., $I = 0.5549$ m^4. Let the loading consist of a dead load of $g = 2$ t. per meter covering the entire span and a live load of $p = 2$ t. per meter covering the left half of the span. Hence $\lambda = \lambda' = \frac{l}{2}$.

By the *approximate theory*, equs. (277) and (278), we obtain

$$H = \frac{1}{8}\left(g + \frac{p}{2}\right)\frac{l^2}{f'},$$

where

$$f' = f\left(1 + \frac{15}{8}\frac{I}{Af^2}\right) = 12.13 \times 1.180 = 13.44 \text{ m}.$$

Hence

$$H = \frac{1}{8} \times 3 \times \frac{\overline{120}^2}{13.44} = 401.79 \text{ t.}$$

The moments are given by $M = \mathbf{M} - Hy$ as follows:

$$\text{For} \quad x = \frac{1}{4}l \qquad \frac{1}{2}l \qquad \frac{3}{4}l$$

$$M = 844.72 \qquad 526.29 \qquad -55.28 \text{ tonne-meters.}$$

For the *exact design,* we first assume the above value for H and thus obtain

$$c^2 = \frac{401.79}{11{,}098{,}000 - 8.287 \times 401.8} = 0.0000362117$$

$$\text{Hence } c = 0.0060176$$

$$\text{and } cl = 0.722112 = 41^\circ\ 22'\ 16''.3.$$

With $\lambda = \frac{l}{2}$, equ. (423) yields

$$H = \frac{432{,}000 + 9{,}941{,}538}{2{,}521.34 + 22{,}332.01 + 970.40 - 5.34} = 401.79 \text{ t.}$$

which is precisely identical with the result of the approximate theory.

Furthermore, with $\lambda = \frac{l}{2}$, equ. (422) yields,

for $x = \frac{1}{4}l$,

$$M = \frac{1}{c^2 \sin cl}\left[\left(g - H\frac{8f}{l^2}\right)\left(\sin\frac{1}{4}cl + \sin\frac{3}{4}cl - \sin cl\right) + p\left(\sin\frac{3}{4}cl + \sin\frac{1}{4}cl,\ \cos\frac{1}{2}cl - \sin cl\right)\right],$$

for $x = \frac{1}{2}l$,

$$M = \frac{1}{c^2 \sin cl}\left[\left(g + \frac{p}{2} - H\frac{8f}{l^2}\right)\left(2\sin\frac{1}{2}cl - \sin cl\right)\right],$$

for $x = \frac{3}{4}l$,

$$M = \frac{1}{c^2 \sin cl}\left[\left(g - H\frac{8f}{l^2}\right)\left(\sin\frac{1}{4}cl + \sin\frac{3}{4}cl - \sin cl\right) + p \cdot \sin\frac{1}{4}cl\left(1 - \cos\frac{1}{2}cl\right)\right].$$

Substituting the numerical values in these expressions,

$$\text{for} \quad x = \frac{1}{4}l \qquad \frac{1}{2}l \qquad \frac{3}{4}l$$

$$M = 872.41 \qquad 556.38 \qquad -39.99 \text{ tonne-meters.}$$

The variation from the approximate theory thus amounts to

	+ 27.69	+ 30.09	+ 15.29 t.-m.
or	3.17%	5.41%	38.2%.

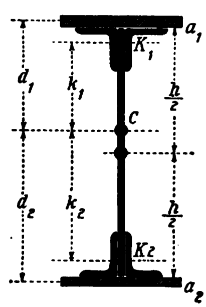
The error involved in the approximate theory, with an arch of adequate stiffness, is thus found to be zero with respect to the horizontal thrust and but a few per cent in the ruling positive moments. At any rate, it should be noted that the accurate theory always yields increased values for the positive moments and that, in a very flexible arch, the error in this respect may be very considerable. However, in all cases of arches applied to bridge construction it will suffice to conduct the design by the approximate theory, especially as the exact theory becomes very complicated when the moment of inertia is variable. If it is desired to calculate the additional moments due to the deformation, an approximate method may advantageously be applied if the arch is not too flexible. This consists in determining the deflections of the arch under the given loading by the rules developed in § § 16, 19 and 25, and then referring the moments to the altered form of the axis. If Δx and Δy are the displacements of an arch-point due to the loading, H the horizontal thrust and V the sum of all the vertical forces on one side of any given section, then the correction to be added to the bending moment on account of the deformation of the axis is given by

$$\Delta \mathbf{M} = V \,.\, \Delta x - H \,.\, \Delta y \quad \text{..................} (424.$$

§ 29. Proportioning of Section in Plate Arch Ribs.

It is difficult to directly determine the exact sectional areas in plate arch ribs, even if the dead load is known or assumed, since the maximum moments cannot be determined without the core-lines and the position of the latter is in turn dependent upon the sectional dimensions and areas. The best we can do is to adopt an approximate method which consists in first estimating the position of the core-points, taking them at about the outer leg of the flange angles.

Fig. 85.



Let the fundamental cross-section consist of a web plate and four equal flange angles (Fig. 85); let h be the depth of this section, a_0 its area, and I_0 its moment of inertia about its gravity axis. Also, let a_1 and a_2 be the required areas of the flange plates, and I the moment of inertia of the combined cross-section $A = a_0 + a_1 + a_2$ whose center of gravity is at the distances d_1 and d_2 from the centers of the flanges. Then we have, very nearly,

$$a_1 d_1 - a_2 d_2 - a_0\left(d_2 - \frac{h}{2}\right) = 0 \quad \text{and } d_1 + d_2 = h.$$

Hence

$$d_1 = \left(1 - \frac{a_1 - a_2}{A}\right)\frac{h}{2}, \quad d_2 = \left(1 + \frac{a_1 - a_2}{A}\right)\frac{h}{2},$$

or, if we write the ratio

$$\frac{a_1 - a_2}{A} = \phi \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (425.$$

then, $d_1 = (1-\phi)\frac{h}{2}$, $d_2 = (1+\phi)\frac{h}{2}$ . . . . (426.

Using these relations, the following approximate expressions for the moment of inertia may be written:

$$I = a_1 d_1^2 + a_2 d_2^2 + I_0 + a_0\left(\frac{h}{2} - d_1\right)^2 = A(1-\phi^2)\frac{h^2}{4} + I_0 - a_0\left(\frac{h^2}{4}\right).$$

If M_1 and M_2 are the maximum moments about the two core-points of the cross-section, (which we have assumed to lie at the flange surfaces of the flange angles), and d'_1 and d'_2 are the distances of the center of gravity of the section from the upper and lower extreme fiber-planes, then the maximum fiber stresses which should equal the permissible intensities of compressive stress are

$$\sigma = \frac{M_2 d'_1}{I} = \frac{M_1 d'_2}{I}.$$

Consequently,
$$\frac{d_1'}{d_2'} = \frac{M_1}{M_2},$$

or, with sufficient approximation,

$$\frac{d_1}{d_2} = \frac{1-\phi}{1+\phi} = \frac{M_1}{M_2},$$

from which we obtain

$$\phi = \frac{M_2 - M_1}{M_1 + M_2} \quad \text{. . . . . (427.}$$

But, since $I\sigma = M_2 d'_1$ or, approximately,

$$\left[A(1-\phi^2)\frac{h^2}{4} + I_0 - a_0\frac{h^2}{4}\right]\sigma = M_2(1-\phi)\frac{h}{2},$$

we have

$$A = \frac{2M_2}{(1+\phi)h\sigma} + \frac{a_0 h^2 - 4I_0}{(1-\phi^2)h^2} \quad \text{. . . . . (428.}$$

or, substituting the value of ϕ from equ. (427),

$$A = \frac{M_1 + M_2}{h\sigma} + \frac{(a_0 h^2 - 4I_0)}{4h^2}\,\frac{(M_1+M_2)^2}{M_1 M_2} \quad \text{. . . . . (429.}$$

The areas of the flange plates may then be calculated by the formulae:

$$\left.\begin{aligned} a_1 &= \frac{1+\phi}{2}A - \frac{1}{2}a_0 = \frac{M_2}{M_1+M_2}A - \frac{1}{2}a_0 \\ a_2 &= \frac{1-\phi}{2}A - \frac{1}{2}a_0 = \frac{M_1}{M_1+M_2}A - \frac{1}{2}a_0 \end{aligned}\right\} \quad \text{. . (430.}$$

This approximate design of the flange sections thus requires that the maximum moments M_1 and M_2 about the outer edges

of the flange angles should be determined. In calculating these moments, however, it will usually suffice to find the critical loading with reference to the center of gravity of the section instead of for the separate core-points. The error introduced by this inaccurate determination of the loading will never be material, since the influence of the loads in the vicinity of the critical points is always comparatively insignificant.* At the same time the design is thus greatly simplified since the number of different cases of loading is reduced by one-half.

With the aid of the cross-sections determined by this first approximation, a more accurate design may then be made in which the true core-lines are located, the maximum moments referred to these, and the effect of temperature taken into consideration. For calculating a_1 and a_2, the above formulae may again be used.

* In *Weyrauch's* design of the Neckar Bridge at Cannstaat, it is shown that the error in the maximum stresses due to the simplified assumption of the loading does not exceed 1.7%. (See *Weyrauch*: "Die Elastischen Bogenträger," 2nd edition, p. 41. Munich, 1897.)

D. Arch and Suspension Systems with Open Web.

(Framed Arches.)

§ 30. Determination of the Stresses in the Members, when the External Forces Are Known.

1. **General.** According to the explanation given at the beginning of this book, the class of bridges under immediate discussion includes all those framed structures which, on account of the occurrence of oblique reactions, may be designated as Arches or Suspension Bridges. In construction, these bridges consist of two chords, at least one of which is curved or polygonal in outline and is connected to the other by a framework or system of web members. In the design, the same fundamental assumptions are made as in the general theory of Trusses, namely that the individual members of the framework are connected at the panel-points with frictionless hinges, so that only axial stresses can appear in the members. The secondary stresses resulting from rigid joints are to be separately considered but, with proper construction, these stresses will as a rule be slighter in arches than in ordinary trusses on account of the smaller deformations; hence it will generally be unnecessary to investigate these stresses in designing an arch. It is further assumed that the deformations are so slight that the initial form of the structure may be made the basis for the computation of the stresses. (Cf. § 28.)

If the external forces, i. e., the applied loads and the resulting reactions, are known, then the calculation of the stresses in the individual members may be executed according to the generally applied rules. If the truss-work itself contains no superfluous members, then the stresses can be simply derived from the equations for static equilibrium; and this may be accomplished:

1. By applying the conditions of equilibrium to the forces acting at each panel-point, which may be done graphically by the method of Cremona's force-polygon; or

2. By the so-called Method of Sections.

The latter method is applicable whenever it is possible to pass a section through the structure so as to cut not more than three non-concurrent members. The stresses in any one of these members may be determined either analytically, by

14

equating the moment of the external forces with that of the unknown internal stress, taking the point of intersection of the other two members as the center of moments; or graphically, by resolving the resultant of the external forces along the directions of the three members in the section. (Fig. 86.)

If the web-bracing contains redundant members, then an exact design requires the application of a procedure based upon a consideration of the elastic deformations of the individual members. There are thus to be obtained as many equations of condition as there are redundant members.

In many cases, however, the procedure may be simplified and the number of statically indeterminate quantities reduced

Fig. 86.

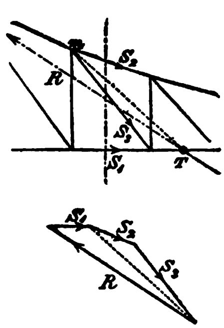
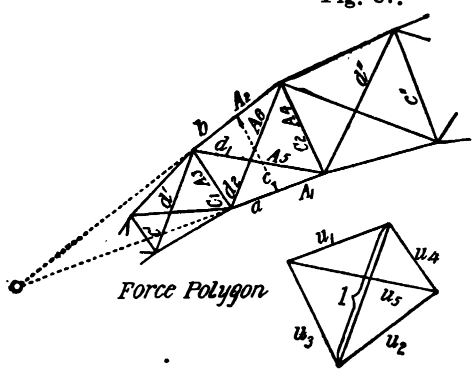


Fig. 87.



by adopting a certain approximation which will here be considered in connection with several systems of web-bracing.

a.) *The framework consists of three sets of web members, all of rigid construction* (Fig. 87). Let the panel-length be relatively small so that the break in direction of consecutive chord members may be inappreciable. Each panel contains a redundant member, viz., one of the two diagonals. If, as is sufficiently accurate, we assume the external reactions to remain unaffected, then the stress in the diagonal will affect no members of the contiguous panel except the radial posts common to both. Introducing the notation indicated in Fig. 87 for the lengths and sections of the members composing any panel, then the stress S_6 of the diagonal d_2 is determined by the following expression of the Theorem of Least Work:

$$\Sigma\, r\, S \frac{dS}{dS_6} = \Sigma\, r\, S\, u = 0 \dots\dots\dots\dots\dots\dots (431.$$

in which the coefficient of elasticity is assumed uniform throughout, r represents the ratio of the length to the cross-section

of each member, and u denotes the stress produced in each member by two opposing unit forces acting in the direction of the member d_2. If $\mathbf{Z}$ represents the stresses which would be produced in the members by the given loading upon the omission of the redundant diagonal, then the actual stress will be

$$S = \mathbf{Z} + S_6 \, . \, u \quad \text{.......................} \quad (432.$$

and only the posts c_1 and c_2 require additional terms, $S' u'$ and $S'' u''$ respectively, to represent the influence of the adjacent diagonals. Accordingly there results from equ. (431),

$$\Sigma\, r\, \mathbf{Z}\, u + S_6 \,\Sigma\, ru^2 + r_3 S' u_3 u' + r_4 S'' u_4 u'' = 0 \quad \text{........} \quad (433.$$

With approximately parallel directions of c_1 and c_2, the stresses u are easily expressible with the aid of the force polygon, Fig. 87, by the following values: (from similar triangles),

$$u_1 = -\frac{a}{d_2},\; u_2 = -\frac{b}{d_2},\; u_3 = -\frac{c_2}{d_2},\; u_4 = -\frac{c_1}{d_2},\; u_5 = \frac{d_1}{d_2}.$$

Furthermore,

$$u' = -\frac{c'}{d'},\quad u'' \quad -\frac{c''}{d''},$$

or, with close approximation,

$$u' = -\frac{c_1}{d_2},\quad u'' = -\frac{c_2}{d_2}.$$

Introducing in the above equation the further approximation $S' = S'' = S_6$, and solving for S_6, there will result the expression

$$S_6 = \frac{\left(Z_1 \frac{a^2}{A_1} + Z_2 \frac{b^2}{A_2} + Z_3 \frac{c_1 c_2}{A_3} + Z_4 \frac{c_1 c_2}{A_4} - Z_5 \frac{d_1^2}{A_5}\right) d_2}{\frac{a^3}{A_1} + \frac{b^3}{A_2} + c_1 c_2 (c_1 + c_2)\left(\frac{1}{A_3} + \frac{1}{A_4}\right) + \frac{d_1^3}{A_5} + \frac{d_2^3}{A_6}} \quad \text{..} \quad (434.$$

No appreciable error will be introduced by using a mean depth of panel c, putting

$$c_1 c_2 = c^2, \qquad c_1 c_2 (c_1 + c_2) = 2\, c^3,$$

and, in addition, assuming an average cross-section A_3 for the posts c_1 and c_2 there will result

$$S_6 = \frac{\left(Z_1 \frac{a^2}{A_1} + Z_2 \frac{b^2}{A_2} + (Z_3 + Z_4) \frac{c^2}{A_3} - Z_5 \frac{d_1^2}{A_5}\right) d_2}{\frac{a^3}{A_1} + \frac{b^3}{A_2} + 4 \frac{c^3}{A_3} + \frac{d_1^3}{A_5} + \frac{d_2^3}{A_6}} \quad \text{..} \quad (434^a.$$

For a final simplification, introduce the closely approximate values

$$Z_3 = Z_4 = -Z_5 \frac{c}{d_1}$$

in the above equation, thus obtaining:

$$S_6 = \frac{\left(Z_1 \frac{a^2}{A_1} + Z_2 \frac{b^2}{A_2}\right) d_2 - Z_5 \left(2 \frac{c^3}{A_3} + \frac{d_1^2}{A_5}\right) \frac{d_2}{d_1}}{\frac{a^3}{A_1} + \frac{b^3}{A_2} + 4 \frac{c^3}{A_3} + \frac{d_1^3}{A_5} + \frac{d_2^3}{A_6}} \quad \text{. . (434}^b\text{.}$$

By equ. (432), the value of S_6 fixes the stresses in all the other members in the panel. Accordingly, the stress in the diagonal d_1 will be

$$S_5 = Z_5 + \frac{d_1}{d_2} S_6$$

or, on substitution,

$$S_5 = \frac{\left(Z_1 \frac{a^2}{A_1} + Z_2 \frac{b^2}{A_2}\right) d_1 + Z_5 \left(\frac{a^3}{A_1} + \frac{b^3}{A_2} + 2 \frac{c^3}{A_3} + \frac{d_2^3}{A_6}\right)}{\frac{a^3}{A_1} + \frac{b^3}{A_2} + 4 \frac{c^3}{A_3} + \frac{d_1^3}{A_5} + \frac{d_2^3}{A_6}}$$

The ratio between the stresses of the two intersecting diagonals will be

$$\frac{S_5}{S_6} = \frac{\left(Z_1 \frac{a^2}{A_1} + Z_2 \frac{b^2}{A_2}\right) d_2 + Z_5 \left(\frac{a^3}{A_1} + \frac{b^3}{A_2} + 2 \frac{c^3}{A_3} + \frac{d_2^3}{A_6}\right) \frac{d_2}{d_1}}{\left(Z_1 \frac{a^2}{A_1} + Z_2 \frac{b^2}{A_2}\right) d_2 - Z_5 \left(2 \frac{c^3}{A_3} + \frac{d_1^3}{A_5}\right) \frac{d_2}{d_1}} \cdot \frac{d_1}{d_2} \quad \text{. . (435.}$$

An approximate value, sufficiently accurate in many cases, may be obtained by neglecting the terms containing A_1 or A_2 as a denominator. There will then result the simple relation

$$\frac{S_5}{S_6} = - \frac{2 \frac{c^3}{A_3} + \frac{d_2^3}{A_6}}{2 \frac{c^3}{A_3} + \frac{d_1^3}{A_5}} \cdot \frac{d_1}{d_2} \quad \text{. . . . . (436.}$$

If the radial posts c are of relatively heavy section when compared with the diagonals, then a final simplification will give,

$$\frac{S_5}{S_6} = - \frac{A_5/d_1^2}{A_6/d_2^2} \quad \text{. . . . . (437.}$$

The expressions (436) and (437) are dependent only upon the lengths and cross-sections of the web members; consequently, if the latter quantities or their ratios be assumed at the outset, the resultant of the two stresses S_5 and S_6 may easily be obtained (Fig. 88). For this purpose, lay off upon the two diagonals from their point of intersection the distances K-1 and K-2 proportional respectively to the numerator and denominator of the above expressions. The diagonal of the resulting parallelogram gives the direction of the resultant of S_5 and S_6 and its intersections o and u with the two chords will be the moment-centers for the determination of the chord stresses. If M_o

and M_u are the moments of the external forces about these two points, p_o and p_u the perpendiculars from o and u upon the lower and upper chords respectively, then the chord stresses will be

$$S_1 = +\frac{M_o}{p_o}, \quad S_2 = -\frac{M_u}{p_u} \qquad (438.$$

b.) *The framework consists of doubly intersecting diagonals, without radial posts.* (Fig. 89.) This system of bracing, in the consideration of its internal stresses, is also statically indeterminate. An approximate determination, sufficient in all cases, may be made by putting $A_3 = 0$ in the equations derived above. There is thus obtained, from equation (435), the following ratio of stresses in two intersecting diagonals:

Fig. 88.

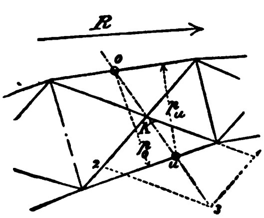
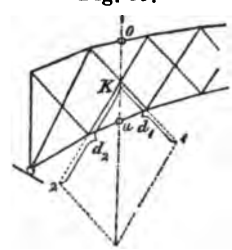


Fig. 89.



$$\frac{S_5}{S_6} = -\frac{d_1}{d_2} \qquad (439.$$

Laying off the distances d_1 and d_2 from the point of intersection K, the diagonal of the resulting parallelogram will cut the chords in the points o and u, about which the moments of the chord stresses are to be taken. The stress in the diagonal will be, by the above equation,

$$S_5 = \frac{1}{2} Z_5, \qquad (440.$$

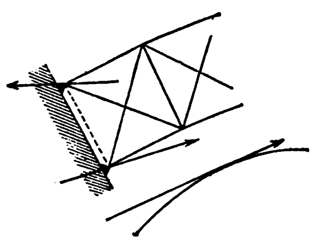
from which may be derived the well-known rule, that the analysis of such a multiple web system may be effected by resolving it into its simple component systems.

c.) *The framework is constructed as in* (a.) *but without rigid diagonals.* The approximate formulae (436) and (437) show that under any given loading the stresses in the two diagonals of any panel will be of opposite character. The exact formula (435), however, may yield a like sign for the two stresses; but, in such case, at least one of these stresses will be very small. It may therefore be assumed that one of the two diagonals in each panel must always remain without

stress. There remains to be determined, under any given loading, merely which of the two diagonals receives a positive **Z**, or a tensile stress; this member is then to be considered as acting alone in its panel, so that the web system is reduced to a statically determinate one.

2. Severest Loading and Determination of Maximum Stresses. If we can find the reactions for any arbitrary position of a concentrated load, then no difficulty will be involved in determining the manner of loading for maximum stress in each member. This may generally be accomplished by the aid of influence lines for the individual stresses, as explained on page 42 of this book. There may also be applied the reaction locus and tangent curves for determining the critical point or load-division point for each stress. In arches with pin-ends, the tangent curves are replaced by the two end-points: and in three-hinged arches, the reaction locus consists of the two lines joining the end-points with the crown-hinge. Arches with double end-connections (Fig. 90), corresponding to arched ribs with fixed ends, may be considered as exerting two reactions at each abutment, the resultant of which coincides with a tangent to the Tangent Curve.

Fig. 90.



The position of continuous loading for producing maximum stress is determined by rules similar to those applying to arched ribs. It is merely necessary to find the center of moments for the member under investigation, and to pass through that point the tangents to the Tangent Curves. For a chord member, the center of moments will be the opposite panel-point or some point determined as in Fig. 88 or 89; for a web member, it will be the intersection of the two chords of the corresponding panel. The intersections of the above tangents with the Reaction Locus constitute the division-points of the loading.

The stresses produced by the loading thus determined may be evaluated by the method of sections or moments, either graphically or analytically, provided the corresponding reactions can be ascertained. Except for the signs of the stresses, there will be no difference in the procedure whether arches or suspension structures are considered.

If the loading consists of a train of concentrations, the maximum stresses are best determined by the aid of influence lines. This procedure, in the case of hinged arches, is to be

especially recommended for its simplicity and will therefore be given a detailed consideration.

3. Method of Influence Lines for Framed Arches with Pin-Ends.

a.) *Determination of the Chord Stresses.* According to equation (438), the upper-chord stress $r\,s$ (Fig. 91) is determined by

$$\underset{(rs)}{S} = -\frac{M_u}{p_u} = -\frac{M_u}{h_u} \cdot sec\,\sigma$$

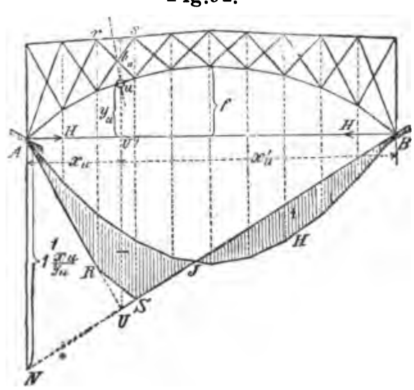
where M_u is the moment of the external forces about the point u (to be determined as above), h_u is the vertical depth of truss at the point u, and σ is the inclination of the upper flange above the horizontal. The moment M_u, however, may be expressed in the case of the two-hinged arch as the moment $\mathbf{M}_u$ of a freely supported beam diminished by the moment of the horizontal thrust, or

$$M_u = \mathbf{M}_u - H\,y_u,$$

so that the chord-stress will be

$$S = -\frac{\mathbf{M}_u - H\,y_u}{h_u} \cdot sec\,\sigma = -\left(\frac{\mathbf{M}_u}{y_u} - H\right)\frac{y_u}{h_u} \cdot sec\,\sigma \,.\,. \quad (441.$$

Fig.91.



If the influence line for the horizontal thrust H, or the "H-curve," is constructed or computed according to methods to be later explained for a moving load = 1, there remains merely to draw an influence line for the expression $\frac{\mathbf{M}_u}{y_u}$,i. e.,

for the moments of a freely supported beam divided by the constant ordinate y_u. This moment influence line, in the case of vertical loads, is known to be a triangle with a maximum ordinate at $u = \frac{x_u x'_u}{l \cdot y_u}$.

Accordingly, if we make the distance $AN = \frac{x_u}{y_u}$ and draw the connecting line NB (Fig. 91), the latter will intercept on the vertical line through the point u a distance $UU' = \frac{x_u x'_u}{l \cdot y_u}$ so that AUB will be the Moment-Triangle. Within the panel, however, all loads must first be transferred to the panel-points r and s, so that the influence line between R and S must be a straight line; consequently the entire moment-triangle will not be effective but the influence line will be represented by $ARSB$. Subtracting from the ordinates of this line those of the H-curve, the differences, multiplied by the numerical factor $\frac{y_u}{h_u} \cdot sec\ \sigma$, will give the stresses in the chord member rs according to equation (441).

For a system of concentrations, the severest position of loading and the corresponding stress may be determined by the general principles of influence lines. For a uniformly distributed load of intensity p per unit length, the largest stresses in the chord rs will be,

$$S_{min} = -\ Area\ ARSJ \cdot \frac{y_u}{h_u} \cdot sec\ \sigma \cdot p,$$

$$S_{max} = +\ Area\ JHB \cdot \frac{y_u}{h_u} \cdot sec\ \sigma \cdot p,$$

in which the areas are to be determined by measuring the abscissae to the scale of lengths and the ordinates to the scale of forces.

If the panel-points of the lower chord lie along a parabola, then it should be noted, for simplifying the construction, that the locus of the points U corresponding to the panel-points of the lower chord will be a horizontal line and its constant ordinate will be $UU' = \frac{l}{4f}$.

b.) *Determination of the Web Stresses.* (Fig. 92.) It will be sufficient to analyze the stresses for a simple web system; and then to make an approximate determination of the stresses in a multiple system either by separating it into simple systems or by eliminating the redundant members with the aid of equations (434) to (435).

The stress in the member rv (Fig. 92) is obtained by taking the moment of the external forces about the point of inter-

section z of the chord members of the corresponding panel, so that

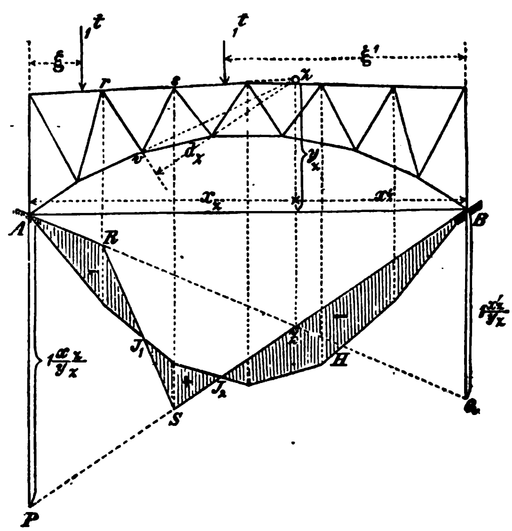
$$\underset{(r\,v)}{S} = + \frac{M_z}{d_z}$$

where d_z represents the perpendicular let fall upon the member from the point z. Again, we may write $M_z = \mathbf{M}_z - H \cdot y_z$, thus obtaining

$$\underset{(r\,v)}{S} = + \frac{\mathbf{M}_z - H\,y_z}{d_z} = + \left(\frac{\mathbf{M}_z}{y_z} - H\right)\frac{y_z}{d_z} \;\ldots\ldots (442.$$

$\mathbf{M}_z$ is again the moment which would exist if the end-points were

Fig. 92.



horizontally movable, or the moment produced in a simply supported beam. For all positions of a unit load ($G = 1$) to the left of the point r, $\mathbf{M}_z = \frac{\xi}{l}\, x'_z$, or equal to the moment of the right reaction about the point z; for the load positions to the right of the point s, the moment of the left reaction will give $\mathbf{M}_z = \frac{\xi'}{l} \cdot x_z$; this affords a simple construction for the influence line of $\frac{\mathbf{M}_z}{y_z}$. For this purpose erect at A a vertical line $A\,P = \frac{x_z}{y_z}$, and at B a vertical line $B\,Q = \frac{x'_z}{y_z}$; the connecting lines $A\,Q$ and $B\,P$, intersecting in the vertical

line through z, determine the polygon $A\ R\ S\ B$ which constitutes the influence line for the function $\frac{M_z}{y_z}$. Deducting the H-ordinates, the shaded area in Fig. 92 will be the influence area for the required web stress; the intercepts of this area, multiplied by the constant factor $\frac{y_z}{d_z}$, represent the stress in the web member $r\ v$ for the different positions of the load. In the case represented, there are two division-points for the loading (critical points); all loading within the limits $J_1\ J_2$ produces a tensile stress, loading in the remaining portions of the span will produce compression in the member. In some cases there may be but one critical point, i. e., but one intersection of the moment polygon with the H-polygon; this will occur when the vertex Z lies outside of the H-polygon.

In the case of a uniformly distributed load p per unit length, the largest stresses in the member $r\ v$ will be

$$S_{\max} = + Area\ J_1 S J_2 \cdot \frac{y_z}{d_z} \cdot p\ ,$$

$$S_{\min} = - (Area\ A R J_1 + Area\ J_2 H B)\ \frac{y_z}{d_z} \cdot p.$$

For an approximate solution in the case of a double web system (Fig. 91), resolve it into its component systems, consider one-half of p acting on each, and apply the method of influence lines as above.

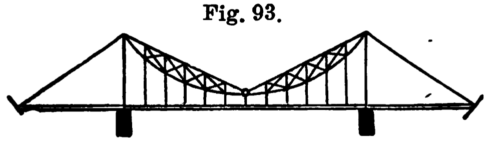
§ 31. The Simple* Framed Arch with Hinged Ends.

Under this head we consider all those framed structures of a single span, which have their ends A and B capable of rotating but not of sliding freely. If, in addition, such a framework is interrupted by a central hinge, there results the statically determinate

a.) Three-Hinged Framed Arch. The equations of static equilibrium suffice to determine the reactions in this type of structure. The reaction locus in this case consists of the two lines passing through the end-points and the crown-hinge; and the reactions are given by equations (200) and (201). The H-polygon for vertical loads reduces to a triangle whose altitude, if the crown hinge is at the middle of the span and if the corresponding rise is denoted by f, is $H = \frac{l}{4f}$.

To the foregoing discussions, nothing need be added concerning the determination of the stresses in the Three-Hinged Arch. These may be found, according to the previous general remarks, either analytically or graphically. The influence lines for chord and web members are constructed as in Fig. 91 and 92, except that the H-polygon is replaced by the triangle already mentioned.

Fig. 93.



The suspension system represented in Fig. 93 is to be considered as an *inverted three-hinged arch* and is to be designed in exactly the same manner. In such a system, if the curve of the cable or chain forming the lower chord is made to coincide with the equilibrium polygon of the dead load, then all the dead load and the main part of all live load that may be uniformly distributed over the entire span will be carried by the lower chord, while the upper chord and the bracing will be stressed only by partial or non-uniform loading.

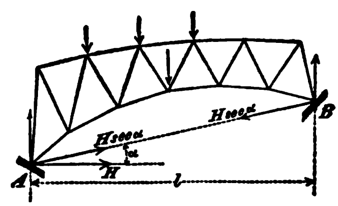
b.) Two-Hinged Framed Arch. As previously stated, this system is statically of single indetermination with reference to

*i. e., having a single span; non-continuous.

the external forces, so that the elastic deformations must be considered in determining the unknown reaction. Various methods of procedure are here applicable.

1. General Equation of Condition for the Horizontal Thrust (H). If one of the ends of the framed arch, e. g., A, were free to slide horizontally, then the system would act as a simple truss for which the internal stresses could be determined in the usual manner. Let these stresses be represented by Z and let u denote the stresses which would be produced by a pair of forces of magnitude $= 1.\ sec\ a$ applied at the end-points and directed toward each other. The stresses u are also easily determined.

Fig. 94.



If, upon anchoring the ends, there is produced a horizontal thrust $= H$, so that the actual force acting along the direction of the arch-chord $= H\ sec\ a$, then the stresses in the arched framework will be

$$S = Z + H\ .\ u \qquad (443.$$

These stresses will produce elastic deformations in the individual members amounting to

$$\Delta s = \frac{s}{EA} S$$

where s is the length and A the cross-section of the member, and E is the coefficient of elasticity. The work of deformation of the internal stresses of the system will be

$$W = \frac{1}{2} \Sigma\ S \cdot \Delta s = \frac{1}{2} \Sigma \frac{S^2 s}{EA} = \frac{1}{2E} \Sigma r S^2,$$

where $r = \frac{s}{A}$ for abbreviation.

Assuming that the abutment is displaced horizontally under loading by an amount Δl, the span l being increased by this amount, then, by a familiar principle concerning the work of deformation (Castigliano's Theorem),

$$-\Delta l . cos\ a = \frac{d\,W}{d\,(H\,sec\,a)},$$

$$\text{or} \quad -\Delta l = \frac{d\,W}{d\,H} = \frac{1}{E} \Sigma r S . \frac{d\,S}{d\,H}.$$

Substituting the value of S from equ. (443) and putting $\frac{dS}{dH} = u$,

$$-E . \Delta l = \Sigma r S u = \Sigma r Z u + H \Sigma r u^2$$

whence

$$H = -\frac{\Sigma r Z u + E . \Delta l}{\Sigma r u^2} \qquad (444.$$

A different analysis gives the value of H in terms of the

virtual deflections at the abutment. Considering the end B as fixed, let the horizontal deflections of the free end A

produced by a unit load at M, be δ_{ma},
produced by a load P_m at M, be $P_m\,\delta_{ma}$,
produced by a force $1 \,.\, sec\,\alpha$ acting along the chord $A\,B$, be δ_{aa},
produced by a similar force $H \,.\, sec\,\alpha$, be $H \,.\, \delta_{aa}$.

The actual horizontal displacement of the point A with reference to the point B, caused by a yielding of the abutments, is denoted by $-\Delta\,l$. This must agree with the resultant effect of all the applied loads P and of the force $H \,.\, sec\,\alpha$, so that

$$-\Delta l = (\Sigma P_m\,\delta_{ma} + H\,\delta_{aa})$$

whence

$$H = -\frac{\Sigma P_m\,\delta_{ma} + \Delta l}{\delta_{aa}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (445.$$

Comparing this with equation (444) shows that $\Sigma r Z u = E \,.\, \Sigma\, P_m\,\delta_{ma}$ and $\Sigma\, r u^2 = E \,.\, \delta_{aa}$; so that the terms of equ. (444) represent the horizontal deflections of the freely supported end of a simple truss, caused respectively by the external loading and a force $1 \,.\, sec\,\alpha$ directed along $A\,B$, multiplied by the coefficient E.

The above determination of the horizontal thrust tacitly assumes an unstressed initial condition of the system; in other words it is presupposed that upon the removal of the external loading the system will remain free from stress. In general, however, this unstressed condition will not obtain except when every member is at that temperature at which it may have been fitted into the structure without initial strain. If the actual state of the individual member differs from this temperature by t^0, then, if ω represents the coefficient of expansion, the equation of condition for H becomes

$$-E \,.\, \Delta l = \Sigma\,(r S + E \,.\, \omega t \,.\, s) u.$$

If it is desired to find the effect of the temperature alone, the external loading may be omitted from consideration; accordingly, putting $Z = 0$,

$$-E.\Delta l = \Sigma E.\omega.t.s.u + H \Sigma r u^2$$

whence

$$H_t = -\frac{E.\Delta l + \Sigma E \omega t s u}{\Sigma r u^2} \quad \ldots\ldots\ldots\ldots (446.$$

It should be observed here that it is impracticable to consider separately the temperature range (t) of each individual member from its unstressed initial condition. It must suffice to assume certain uniform temperature limits for all the members

or at least for groups of members, although this makes it uncertain whether the severest combination of conditions is provided for. It is therefore important that the range of temperature, at any rate, should not be assumed too small. Under the conditions of our climate, there should be assumed a value of $t = \pm 30°C.$ $(= \pm 54°F.)$; furthermore, it is easily practicable to apply the above formula to the case of the unequal heating of the two chords occurring when one of these is shielded from the sun.

Assuming ωt to be the same for all the members, then $\Sigma \omega t s u = \omega t \Sigma su$. There is a theorem, first demonstrated by *Mohr**, as follows: If, in a free, unloaded structure, two unit forces directed toward each other are applied at any two panel-points $A B$ separated by a distance s_1, the resulting stresses u' produced in the various members s will satisfy the condition

$$\Sigma u' s + s_1 = 0.$$

In the present case, the distance between the panel-points $A B = l . sec\, \alpha$, and u represents the stresses produced in the framework by the forces $1 \; . \; sec\; \alpha$ acting along the chord $A B$; consequently $u' = u \cos \alpha$ and $\Sigma us = - l . sec^2 \alpha$, thus giving

$$H_t = - \frac{E(\Delta l - \omega t l \sec^2 \alpha)}{\Sigma r u^2} \quad \ldots\ldots\ldots\ldots\ldots (447.$$

For steel we may use $E \omega = 250$ tonnes per square meter per degree C. (= 197.5 pounds per square inch per degree F.), so that for $t = 30°C.$ $(= 54°F.)$, $E \omega t = 7{,}500$ tonnes per square meter (= 10,660 pounds per square inch).

It may be observed here that the above derivation of H is not exact inasmuch as it assumes, in determining the stresses, that the initial geometric form of the system remains unchanged. Only under this assumption can the theorem of virtual work be applied. This is, however, the same approximation as was made for the theory of the Arched Rib, and the same remarks apply concerning its permissibility as have been made in § 28. This method of analysis becomes better applicable to the framed arch as its radial depth increases and as its deflections, in consequence, diminish.

2. Analytical Determination of the *H*-Influence Line. In order to obtain the influence line for the horizontal thrust H, it is necessary to consider a unit load placed successively at the individual panel-points and to compute the corresponding values of $\Sigma r Z u$. In a symmetrical arch, this computation is most easily accomplished by finding the stresses u and z produced in each of the members of the framework by a horizontal and vertical unit reaction applied at the end A. The stresses u and z may be determined either analytically or graphically, the force polygon being used in the latter procedure.

*Zeitschrift d. Arch. u. Ing. Ver. zu. Hannover, 1874.

If, now, a load of unity be placed at any panel-point whose horizontal distances from the ends A and B are x and x' respectively, then the horizontal thrust is given by the formula

$$H_x = -\frac{l\sum\limits_0^x rzu + x\sum\limits_x^{x'} rzu}{l\sum\limits_0^l ru^2} \quad \ldots\ldots\ldots\ldots\ldots (448.$$

The summation $\sum\limits_0^x$ refers to all the members between the end A and the load, and the summation $\sum\limits_x^{x'}$ refers to all the members between the point of application of the load and the panel-point symmetrical thereto. If the panel lengths are all equal, the distances x, x' and l may be replaced by the corresponding numbers of panels.

In an asymmetrical arch, an additional force polygon, z', is to be constructed, viz., for a vertical load of unity applied at the end B; and the numerator of the expression for H must be changed to $x'\sum\limits_0^x r\,z\,u + x\sum\limits_0^{x'} r\;z'\,u$.

3. Graphical Determination of the H-Influence Line. The effect of a concentration P is given by equation (445) as $H = \frac{P\,\delta_{ma}}{\delta_{aa}}$. The expressions δ_{ma} and δ_{aa} may also represent the projections upon the chord $A\,B$ of the horizontal deflections of A produced respectively by the load $P = 1$ and by the force $(1\,.\,sec\,\alpha)$ acting along the line $A\,B$. According to Maxwell's Theorem, however, the deflection of the point A along the direction of the chord caused by the vertical unit load, must be equal to the vertical deflection of the loaded panel-point that would be produced by a unit force acting along the chord. Consequently the influence line for H is given by the vertical deflections of the panel-points (elastic curve of the loaded chord) produced by two opposing unit forces acting along the chord $A\,B$. The scale unit for H is the relative displacement of the points A and B produced by two opposing forces of $(1\,.\,sec\,\alpha)$ acting along the same chord $A\,B$. For evaluating these displacements, various exact or simplified procedures may be followed.

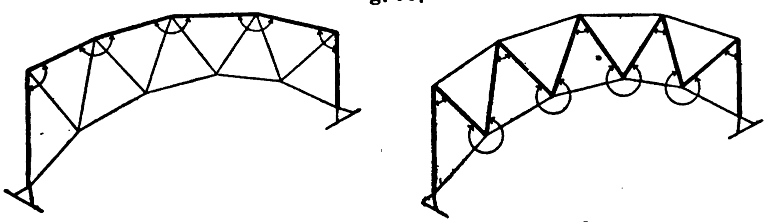
a.) *Method of the Williot Deflection Diagram.* (Plate 1, Figs. 2-2^c.) We first construct a Cremona Force Polygon for the stresses u_1 produced by two unit forces acting along $A\,B$ (Fig. 2^a). From these values and the known lengths and cross-sections of the members, are computed the elongations multiplied

by E: $E.\Delta s = \frac{s}{A}.u_1$. The latter quantities are used for the construction of a Williot Diagram (Fig. 2[b]). For this purpose we consider any arbitrary member (kd in Fig. 2[b]) as fixed and lay off its elongation (here $= 0$) from a pole k_1 to the point d_1. From k_1 d_1 are drawn the elongations of the contiguous members ki and di *in their respective directions* (a lengthening being drawn in the direction of the member, a shortening in the opposite direction) and at the ends of these two lines are erected perpendiculars thereto, representing the angular displacements of the members. The intersection i_1 of these perpendiculars represents the change in position of the panel-point i relative to kd. Proceeding thence in the same manner, the elongations of the members ic and dc fix the position of c_1, and so on to A_1. Proceeding similarly on the other side of kd, we obtain the point l_1, then e_1, and finally the point B_1, thus determining the relative displacement of the points A and B. Since A, however, is constrained to move in a horizontal plane, there must be effected such a rotation of the entire framework as will eliminate the difference of elevation of the points A_1 and B_1. In this rotation about B, A will move in a direction perpendicular to $A B$ through the distance $B_1 A_2$, and the diagram of displacements corresponding to this rotation consists of the figure $A_2 b_2 c_2 \ldots$, geometrically similar to the actual framework; finally, therefore, the distance between correspondingly designated points of the two displacement diagrams will give the actual deflections of the respective points b, c, etc. If the moving concentration G is applied only at the panel-points g, h, i, etc., of the upper chord, then the *vertical* displacements of these points, scaled from the Williot diagram, will represent the ordinates of the H-influence line, all upward deflections corresponding to positive values of H. These values are plotted in Fig. 2[c] to one-half the scale of the displacement diagram. The scale unit for H or the value of the corresponding concentration G is therefore $\frac{1}{2}\,\delta_{aa}$, where δ_{aa} is the displacement of A in the direction $A B$ produced by a force $1 . \sec \alpha$ acting in the same direction, or, what is equivalent, the displacement produced by a unit load multiplied by $\sec \alpha$. This quantity is represented by the horizontal displacement $A_1 A_2$ to be scaled from the Williot Diagram.

b.) *Construction of the Deflection Curve* (*H-Influence Line*) *as the Funicular Polygon of the Deformations of the Angles.* According to the above discussion, the influence line for H coincides with the deflection curve (elastic curve) of the loaded chord produced by the action of a unit force directed from A toward B. This deflection curve may be derived from the

angular and linear deformations of a chain of members connecting the panel-points in question, which chain may consist either of chord members or web members (as indicated by the

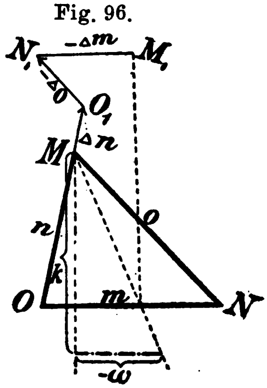
Fig. 95.



a. b.

heavy lines in Fig. 95a and Fig. 95b respectively).

Fig. 96.



The changes in the angles of this linkwork will produce vertical deflections of the panel-points; these deflections may be obtained as the ordinates of a funicular polygon constructed by treating the angular deformations as loads concentrated at the respective panel-points.

The angular distortions may be determined either analytically or graphically. If $M\,N\,O$ (Fig. 96) is one of the triangular frames connected with the point M of the linkwork, and if λ_m, λ_n, λ_o are the longitudinal strains in the bars composing it, then the angular distortion at M will be

$$\Delta\,(n\,o) = (\lambda_m - \lambda_n)\,cotan\,(m\,n) + (\lambda_m - \lambda_o)\,cotan\,(m\,o).$$

The same quantity may be found graphically by constructing, to any convenient scale the vector summation $M\,O_1\,N_1\,M_1$ of the strains in the triangular frame $M\,N\,O$, (all elongations being plotted in the direction of the cyclical succession of the sides of the triangle M-N-O, and all compressions being plotted in the reverse direction); the change in any angle will then be represented by the projection of this chain of lines upon the opposite side of the triangle. Adopting an arbitrary radius k for the angular scale-unit, then the distance w, measured to the linear scale of the plotted strains and divided by k, will represent the actual change in the angle in radians. Note that w is positive if it is directed from the perpendicular line through M in the positive cyclical direction, $M\,N\,O$. In the link-chain of Fig. 95a, the changes in all the angles meeting at a panel-

point must be added algebraically to get the total angular distortion at the point; in the arrangement of Fig. 95b, it is necessary to consider only those angles formed by the web members themselves.

Construct the force polygon of the distances w with a pole distance p; then, considering these quantities as vertically applied loads, construct the corresponding funicular polygon; the ordinates of this polygon, measured from the closing side, will represent, to a scale of $\frac{k}{p}$ times that of the plotted elongations, the deflections due to the angle-changes. The polygon will therefore be an influence line for H. In similar manner we may represent the denominator δ_{aa} of the expression for H, i. e., the displacement of A in the direction AB produced by a force $1 . \sec \alpha$. The part of this displacement due to the angular deformation will be given by the intercept upon the chord AB of a funicular polygon constructed for the quantities w considered as forces acting parallel to AB, with a pole-distance $p. \cos \alpha$ (since the quantities w were determined for a force of unity).

In Plate I, Figs. 3 to 3^e, this procedure is shown applied to the same structure as that used in Fig. 2. The linkwork was formed of the web members according to the scheme of Fig. 95b. The quantities w were obtained graphically from the elongations in the table (column 5) which were plotted to the same scale as that of the Williot Diagram (Fig. 2^b). All the values of w were of the same sign (negative), indicating that all the displacements were upward; this corresponds to positive values for H. Furthermore, having assumed $k = 5$ meters and the pole distance $p = 10$ meters, the funicular polygons Figs. 3^c and 3^d give the displacements to a scale of $\frac{k}{p} = \frac{1}{2}$ times that of the elongations, i. e., to the same scale as that of Fig. 2.

To the deflections caused by the angular deformations must be added those produced by the pure elongations of the members in the chain. These are scaled from a simple Williot Diagram (Fig. 3^e), obtained by the vector addition of the elongations of the members, to which is attached a small reproduction of the structure to provide for the rotation of the system back to its horizontally constraining abutment. The vertical deflections of the panel-points of the framework are taken from this diagram and (reduced to $\frac{1}{2}$ scale) are annexed to the Deflection Curve (Funicular Polygon, Fig. 3^d); similarly, the corresponding horizontal displacement δ''_{aa} of the point A is added to the displacement δ'_{aa} obtained in Fig. 3^c from

the angular distortions. We thus obtain the quantity $\frac{1}{2}\delta_{aa} = \frac{1}{2}(\delta'_{aa} + \delta''_{aa})$ as the magnitude of the load G which produces the values of H given by the ordinates of Fig. 3^d; G thus constitutes the scale or unit-load of the H-influence line.

The two procedures a) and b) yield identical results; in general, method b) permits of securing a somewhat greater degree of graphic precision.

c.) *Simplified Method for the Determination of H.* The determination of the horizontal thrust H will be substantially simplified if the elongations of the web members be neglected and only the effect of the chord members considered. The web members have relatively but a small share in the total deformation of the system, so that this simplification will usually be within the permissible limits of accuracy. It can be introduced in the analytical determination of H (paragraph 2.), as well as in the graphical procedures (a) and (b). It is thus necessary to find the stresses u and the resulting elongation terms $\frac{s}{A}u = r \cdot u$ merely for the chord members. The chord stresses u (produced by the force 1 . $sec\ \alpha$) may be obtained either from a force polygon or by computation. Consider the chord member lying opposite to the panel-point m (Fig. 97); let σ_m be its inclination from the horizontal, y_m the vertical height of the point m above the arch-chord, and h_m the vertical depth of truss; then the stress in the member will be

$$u_m = \frac{y_m}{h_m} sec\,\sigma_m \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (449.$$

Again applying method (b) on the basis of a chain of links consisting of the web members, the weight w, to be applied at m and representing the angular distortion, is given by $w = ru \cdot \frac{k}{d}$ where d is the perpendicular distance of m from the chord member and k is an arbitrary constant. Putting $k = 1$ and $d = h_m\,cos\,\sigma_m$, we have

$$w_m = \frac{s_m}{A_m} \cdot \frac{u_m}{h_m} sec\,\sigma_m,$$

or, with a_m as the length of the horizontal projection of the chord member,

$$w_m = \frac{a_m}{A_m} \cdot \frac{u_m}{h_m} \cdot sec^2\,\sigma_m \quad \ldots\ldots\ldots\ldots\ldots\ldots (450.$$

Introducing an arbitrary standard section A_c, the quantities

w may be replaced by the new numbers $v = A_c \cdot w$, whence

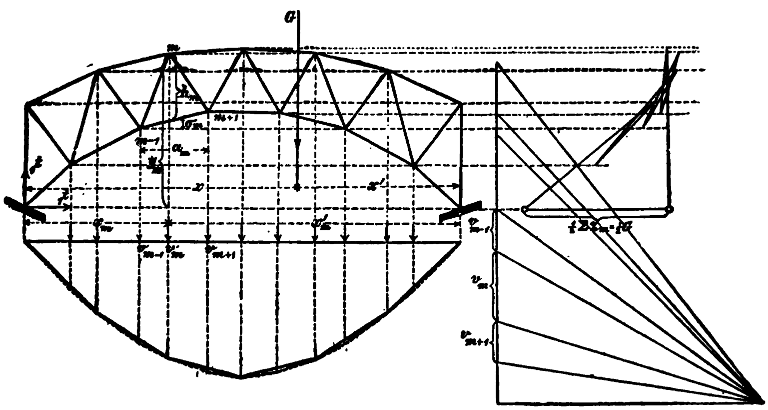
$$v_m = \frac{A_c}{A_m} \cdot \frac{a_m}{h_m} u_m . sec^2 \sigma_m = \frac{A_c}{A_m} \cdot \frac{a_m y_m}{h^2_m} . sec^3 \sigma \text{ .... } (451.$$

We can now obtain the ordinates for the H-influence line as the moments $\mathbf{M}_v$ producible in the framework when acting as a simply supported beam and loaded vertically at the panel-points with the weights v_m; while the unit load G will be given by $\Sigma v_m y_m$; accordingly,

$$H = \frac{\mathbf{M}_v}{\Sigma v y} \text{ .... } (452.$$

This relation could also have been obtained from equ. (448).

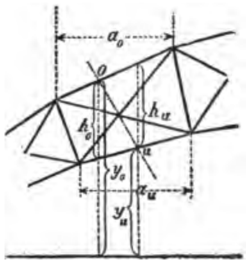
Fig. 97.



The signs of the quantities v need no further consideration as we have already seen that the angular deformations w will have the same sign for all the panel-points of the upper and lower chords.

The graphic method again involves the construction of two funicular polygons, one for the forces v acting vertically and a horizontal pole distance p', the other for the forces v directed parallel to A B with a vertical pole distance p' (Fig. 97). In Plate I, Figs. 4^a-4^c, this simplified determination of H is carried out for the framed-arch treated in the preceding illustrations and, for comparison, the resulting influence line for H is introduced in Fig. 3^d in dash and dot lines. For this purpose, the pole distance p' was so chosen as to make the funicular polygon Fig. 4^c yield an intercept identical with the distance $\frac{1}{2}$ δ_{aa} of Fig. 3, so that the same scale unit G will apply to the H-polygon. The discrepancy with the exact determination of H amounts to .15% in the case considered.

Fig. 98.



If the loads in the arch act only at the panel-points of *one* of the chords, then the H-influence line will have in common with the funicular polygon of v_m only those vertices which lie under the panel-points of the loaded chord (Fig. 97). If the framework contains vertical web members, then the values of v_m belonging to panel-points in the same vertical line must of course be added together (Plate I, Fig. 4). If the framework consists of radial posts and intersecting diagonal struts (Fig. 98), then instead of the panel-points we must use the points o and u determined by the method of Fig. 88; for these points, if the chord-sections are A_o and A_u respectively, the quantities v must be calculated as follows:

$$\left.\begin{aligned} v_o &= \frac{A_c}{A_u} \cdot \frac{a_u y_o}{h_o^2} \cdot \sec^3 \sigma_u \\ v_u &= \frac{A_c}{A_o} \cdot \frac{a_o y_u}{h_u^2} \cdot \sec^3 \sigma_o \end{aligned}\right\} \qquad (453.$$

The same applies to structures consisting of simply intersecting diagonals. If h, the depth of the framed arch, is small compared to the rise, and if the cross-sections of the upper and lower chords are put equal to each other (compare subsequent remarks on this matter), then it will generally be sufficient to use the approximation

$$v_o + v_u = 2\frac{A_c}{A_m} \cdot \frac{a_m y_m}{h_m^2} \cdot \sec^3 \sigma_m \qquad (454.$$

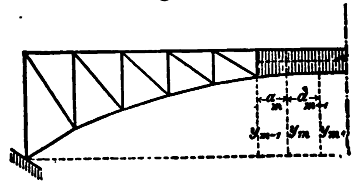
where y_m, h_m, a_m, $\sec \sigma_m$ refer to the axis of the arch and to the point of intersection of the two diagonals. This load, $v_o + v_u$, is then to be considered as acting in the vertical line through this crossing-point.

The horizontal thrust produced by a displacement of the abutments of Δl and a uniform change in temperature will be, by equ. (447),

$$H_t = -\frac{(\Delta l - \omega t l \sec^2 a) E A_c}{\sum\limits_0^l v y} \qquad (455.$$

In the braced arch with straight upper chord, the bracing near the crown is commonly replaced by a full plate web (Fig. 99). For this central portion, equ. (250) of the theory of the Plate-Arch-Rib may be used for calculating the horizontal thrust; if A_b is the

Fig. 99.



average cross-section of the arch-rib and s the length of its axis, the resulting expression for the horizontal thrust will be

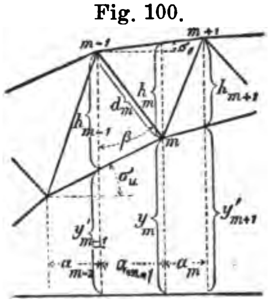
$$H = \frac{M_v}{\Sigma y v + \Sigma y_m v_m + s \frac{A_c}{A_b}} \quad \ldots\ldots\ldots\ldots\ldots (456.$$

Here $\Sigma y v$ refers to all the panel-points of the framed portion, the values of v being calculated by formula (451); $\Sigma y_m v_m$, on the other hand, refers to successive sections of the plate-rib, y_m representing the ordinate of the axis of the rib and v_m being figured by equation (255), viz.,

$$v_m = \frac{a_m}{6} \cdot (2y_m + y_{m-1}) \frac{A_c}{I_m} + \frac{a_{m+1}}{6} (2y_m + y_{m+1}) \frac{A_c}{I_{m+1}} \quad \ldots . (457.$$

M_v is again the moment produced by loads of magnitude v acting on a simple beam of span l; so that, as before, the influence line for H will appear as the funicular polygon of the v-forces.

Fig. 100.



d.) *Correction of the Preceding Method of Determining H by Introducing the Effect of the Web Members.* Pass a vertical section through the diagonal d_m (Fig. 100); then, for a vertical end-reaction of unity, the condition of static equilibrium requires the following relation between the horizontal components of the stresses in the three members cut by the section:

$$z_d \sin\beta = -z_{m-1} \cos\sigma_u - z_m \cos\sigma_o,$$

or
$$z_d = \left(-\frac{M_{m-1}}{h_{m-1}} + \frac{M_m}{h_m}\right)\frac{d_m}{a_{m-1}} = \left(-\frac{x_{m-1}}{h_{m-1}} + \frac{x_m}{h_m}\right)\frac{d_m}{a_{m-1}};$$

for a horizontal end-reaction of unity,

$$u_d = -(1 + u_{m-1} \cos\sigma_u + u_m \cos\sigma_o)\frac{d_m}{a_{m-1}} = \left(\frac{y'_{m-1}}{h_{m-1}} - \frac{y_m}{h_m}\right)\frac{d_m}{a_{m-1}}$$

If A_d is its cross-section, then the contribution of the diagonal d_m to the summations Σrzu and Σru^2 of equ. (448) will be given by

$$rzu = \frac{1}{A_d} \cdot \frac{d^3_m}{a^2_{m-1}}\left[-\frac{x_{m-1}}{h_{m-1}}\left(\frac{y'_{m-1}}{h_{m-1}} - \frac{y_m}{h_m}\right) + \frac{x_m}{h_m}\left(\frac{y'_{m-1}}{h_{m-1}} - \frac{y_m}{h_m}\right)\right]$$

$$ru^2 = \frac{1}{A_d} \cdot \frac{d^3_m}{a^2_{m-1}}\left[\left(\frac{y'_{m-1}}{h_{m-1}} - \frac{y_m}{h_m}\right)\frac{y'_{m-1}}{h_{m-1}} - \left(\frac{y'_{m-1}}{h_{m-1}} - \frac{y_m}{h_m}\right)\frac{y_m}{h_m}\right].$$

Noting that exactly similar expressions may be obtained for the diagonal d_{m+1} (ascending toward the right), whose cross-

section may be denoted by A_d', a comparison with equ. (451) will show that the influence of the elongations of the web-members may be taken into consideration by augmenting each panel load, v_m, at the mth panel-point by an amount:

$$\Delta v_m = \frac{A_c}{A_d} \cdot \frac{d^3_m}{a^2_{m-1} h_m} \left(\frac{y_m}{h_m} - \frac{y'_{m-1}}{h_{m-1}} \right) + \frac{A_c}{A_d'} \cdot \frac{d^3_{m+1}}{a^2_m . h_m} \left(\frac{y_m}{h_m} - \frac{y'_{m+1}}{h_{m+1}} \right) \quad \ldots (458.$$

and at the $(m-1)$th panel-point:

$$\left. \begin{aligned} \Delta v_{m-1} = \frac{A_c}{A_d} \cdot \frac{d^3_m}{a^2_{m-1} . h_{m-1}} \left(\frac{y'_{m-1}}{h_{m-1}} - \frac{y_m}{h_m} \right) \\ + \frac{A_c}{A_d''} \cdot \frac{d^3_{m-1}}{a^2_{m-1} h_{m-1}} \left(\frac{y'_{m-1}}{h_{m-1}} - \frac{y_{m-2}}{h_{m-2}} \right) \end{aligned} \right\} \quad \ldots . (459.$$

In addition, the denominator $\Sigma\, vy$ must be increased by $\Delta v_m . y_m$ and $\Delta v_{m-1} . y'_{m-1}$, respectively.

These formulae are easily modified for the case in which one set of web members are vertical. ($a_{m-1} = a_{m+1} = \ldots . = 0$).

4. **Tentative Estimate of Cross-Sections.** In evaluating H it is necessary to know the areas of cross-sections of the individual members. As these, however, remain to be determined by the design, we are compelled to provide for a recomputation; and, for the first determination of H, we must make a preliminary estimate of the cross-sections of the members. As shown above, the chords have a considerable effect upon the magnitude of the deformations while the influence of the web members is much less. It will therefore be advisable first to determine H by the simplified method in which the elongation of the web members is neglected; or else, if it is desired to include the effect of these members, a uniform section may be assumed for all of them. The assumption of uniform sections may even be admissible for the chord members, in a preliminary computation. Designs actually carried through have shown that the error of such assumptions is negligible, particularly in arches having the hinges in their neutral axis and with chords more or less parallel, as well as in arches of crescent shape. Arches with straight upper chord or those having their end-hinges in the intrados, will of course display a larger variation in chord-section. In order to obtain the closest values of H by the first computation in such cases, it is advisable first to determine the chord-sections as those of a three-hinged arch whose crown-hinge is somewhat above the mid-point between the two chords. If we assume a constant cross-section for the upper chord and another for the lower, then only the ratio between these values will be required in the expression for H_x; the mean cross-section, however, must be known or assumed in evaluating H_t. Observing, furthermore, that the elongations in the vicinity of the crown of the arch are most effective in influencing the value

of H, we should therefore use that ratio of cross-sections which will actually obtain near the crown. Various values may, of course, be adopted for this ratio, but only a certain particular value will yield equal intensities of stress in both chords. Actual designs have shown that in framed arches with either parallel or non-parallel chords, subjected to a moving load, the most effective value for the ratio between the chord-sections at the crown is approximately unity. It is therefore advisable, for a first design, to assume equal sections for the two chords at the crown; and, in arches having their end-hinges in the neutral axis, the chord-sections may be assumed uniform throughout.

In an arched truss subjected to a moving load, as in all other statically indeterminate structures, it is possible to determine the position of loading for maximum stress in each member and thus obtain directly, without trial, the required cross-sections so that all parts of the structure shall be proportioned for equal intensities of stress. Such a design, however, would not only be impractically laborious, but would also be of doubtful value since there are many other sources of error such as non-uniform elastic coefficients, erection stresses and uncertainties in the temperature stresses. We will therefore always employ the above indicated approximate method based on a provisional assumption of cross-sections and, if necessary, go through a recomputation based on the results of the preliminary design.

It is a different matter, however, with an arched truss to be designed for a fixed position of loading. In such a case, as demonstrated in the paper cited below,* it will not, in general, be possible to secure equal intensities of stress in all parts of the structure, except with such perfect erection as is very rarely attained. Such structures may be designed by a simpler method which consists in so choosing the stresses $\pm\sigma = \frac{S}{A}$ for the individual members as to satisfy the equation of condition $\Sigma \frac{S}{A} \cdot \frac{dS}{dH} \cdot s = \Sigma\ (\pm \sigma u s) = -E\,\Delta\, l$. The provisional assumption of cross-sections will therefore be superfluous; however, as the value of the statically indeterminate reaction or of the stress in one of the redundant members may here be assumed arbitrarily within certain limits, namely so as to exclude any consequent change in the sign of the stresses, it will be possible to obtain a series of differently proportioned structures which shall all be stressed to satisfy the above condition. (Compare the previous citation.)

5. Computation of Deflections. In the framed arch, let a load P be applied at any panel-point C whose horizontal distances from the two abutments are ξ and ξ'. Let H_ξ be the horizontal thrust producible by a load of unity at C. In order to calculate the consequent deflection Δy of a panel-point D, distant x and x' from the two abutments, let us consider a unit load applied at D and determine the resulting stresses z_x in the various members of the statically determinate, i. e., freely supported, truss. If the stresses in the framework pro-

* "Beitrag zur Berechnung statisch unbestimmter Stabsysteme." Zeitschr. d. österr. Ing. und Arch. Ver., 1884, No. 3.

duced by the load P at C are denoted by $S = P\ (z_\xi + H_\xi \cdot u)$, so that $\frac{r}{E} \cdot S$ represents the elongations of the individual members, then the principle of equality between external and internal virtual work will give the equation

$$\Delta y . 1 = \Sigma \frac{r}{E} S . z_x = \frac{P}{E} [\Sigma r z_\xi z_x + H_\xi \Sigma r u z_x]$$

which may be rewritten either as

$$\Delta y = \frac{P}{E} \left[\frac{1}{H_\xi} \Sigma r z_\xi z_x + \Sigma r u z_x \right] H_\xi \text{ .... (460.}$$

or, since $\Sigma\ r u z_x = - H_x\ \Sigma\ r u^2$, as

$$\Delta y = \frac{P}{E}\ [\Sigma r z_\xi z_x - H_\xi H_x \Sigma r u^2] \quad \text{........(460}^a.$$

The latter form will be suitable for computation if the thrusts H_ξ and H_x, which are produced by unit-loads at C and D, have already been determined by equ. (448) or in any other way. The summation $\Sigma r z_\xi z_x$ may likewise be computed in terms of the series of stresses z, which are produced in the members of the framework by an upward unit reaction at the end A, by means of the formulae:

$$\left.\begin{array}{ll} \text{for } \xi < x, & \Sigma r z_\xi z_x = \frac{x' \xi'}{l^2} \sum\limits_0^\xi r z^2 + \frac{x' \xi}{l^2} \sum\limits_\xi^x r z^2 + \frac{x \xi}{l^2} \sum\limits_x^l r z^2, \\ \text{for } \xi > x, & \Sigma r z_\xi z_x = \frac{x' \xi'}{l^2} \sum\limits_0^x r z^2 + \frac{x \xi'}{l^2} \sum\limits_x^\xi r z^2 + \frac{x \xi}{l^2} \sum\limits_\xi^l r z^2. \end{array}\right\} \text{..(461.}$$

In a completely loaded span, the crown deflection may be found with sufficient accuracy from the average of all the panel deflections. If the panels are uniform in length, if P is the load per panel-point of which there are n, and if S denotes the stresses produced in the system by the application of a unit load at every panel-point, then, on the assumption of a parabolic deflection polygon, the crown displacement will be given by equating the external and internal work as follows: (Here S and z are identical.)

$$\Delta y_0 = \frac{3}{2} \frac{P \Sigma r S^2}{n . E} \quad \text{..........................(462.}$$

The horizontal deflection Δx of the panel-point D (positive if inwards) produced by a vertical load P applied at C is given by the principle of virtual work as

$$1 . \Delta x = \Sigma \frac{r}{E} \cdot S_\xi u',$$

or

$$\Delta x = \frac{P}{E}\ [\Sigma r z_\xi u' + H_\xi\ \Sigma r u u'] \text{ ....(463.}$$

Here u' denotes the stresses produced in the members of the

simply supported structure by a horizontal force of unity, directed inwards, applied at D. These stresses, again, may be determined either analytically or by means of a force polygon.

The following equations will serve to determine the displacements produced by temperature changes and the simultaneous yielding of the abutments. The vertical deflection of the panel-point D will be

$$\Delta y_t = \Sigma z_x\left(\omega t s + \frac{r}{E} H_t u\right) = \omega t \Sigma s z_x + \frac{H_t}{E} \Sigma r u z_x.$$

Introducing the value of H from equ. (446),

$$\Delta y_t = \omega t \Sigma s z_x - \frac{\Delta l + \omega t \Sigma s u}{\Sigma r u^2} \cdot \Sigma r u\, z_x = \omega t \Sigma s z_x - (\omega t \Sigma s u + \Delta l) H_x.$$

Here s denotes the length of the individual members, and H_x is the horizontal thrust produced by a unit load acting at D.

According to Mohr's Theorem, cited on page 222, we have $\Sigma su = -l \sec^2 a$, also $\Sigma s z_x = 0$, so that the formula for temperature deflections becomes

$$\Delta y_t = \frac{H_t}{E} \Sigma r u z_x = \pm (\omega t l \sec^2 a - \Delta l) . H_x \quad \ldots . (464.$$

The horizontal displacement of the panel-point D on account of temperature changes will be,

$$\Delta x_t = \Sigma u'\left(\omega t s + \frac{r}{E} H_t\, u\right) = \omega t \Sigma u' s + \frac{H_t}{E} \Sigma r u u'$$

or

$$\Delta x_t = \pm \omega t \Sigma u' s \pm (\omega t l \sec^2 a - \Delta l) \frac{\Sigma r u u'}{\Sigma r u^2} \quad \ldots\ldots\ldots (465.$$

In the graphic procedure, our design will be to determine the influence lines for the horizontal and vertical displacements of an arbitrary panel-point D. These influence lines (by Maxwell's Theorem) represent the sag-diagrams or elastic curves of the framework for a vertical or horizontal load applied at D.

Each of these elastic curves may be obtained:

a.) By drawing the Williot Displacement Diagram. For this purpose it is necessary to determine the stresses in the members, produced by a vertical or horizontal unit load applied at D, and to calculate the resulting elongations.

b.) From the funicular polygon of the angle-changes considered as vertical loads. Here the angle-changes may either be determined directly from the strains in the members of the structure or else they may be taken as the resultant of the angle-changes in the freely supported truss and those produced by the force $H_x . \sec a$ acting along the chord AB. The latter quantities yield an elastic curve which coincides with the H-influence line; and, in fact, if the construction described in

method 3.b) is followed, these (negative) deflections will be given by the ordinates of the *H*-curve if they are measured to a scale whose unit is $\frac{k}{p \, . \, H_x \, sec \, \alpha}$ times that of the scale to which the strains were plotted. It therefore suffices merely to compute the *elastic curve for the simple truss* loaded at *D*. For this purpose, we must compute the stresses z_x and the resulting changes in length and angle of the chain of members; laying off the angle-changes upon a radius *k*, the resulting lengths *w* are to be treated as vertical forces applied at the panel-points. For these, construct a funicular polygon with a pole distance of $p \, . \, H_x \, sec \, \alpha$ and correct the resulting ordinates by the small vertical deflections which correspond to the elongations of the members multiplied by $\frac{1}{H_x \, . \, sec \, \alpha}$, i. e., to $r \, z_x \, . \, \frac{1}{H_x \, sec \, \alpha}$, and which are obtained by means of a Williot diagram (similar to Fig. 3[e], Plate I). The differences between these corrected ordinates of the funicular polygon and the ordinates of the *H*-influence line will then give the actual vertical deflections to a scale of $\frac{k}{p \, . \, H_x \, sec \, \alpha}$ times that to which the strains were plotted.

If the simplified method 3.c) has been used for obtaining the *H*-curve, the latter being constructed as the funicular polygon of the panel loads v_m (equ. 451), then in determining the elastic curve of the simple truss there again need be considered the strains in the chord members only. The loads v'_m, proportional to the angle-changes in the linkwork, are then to be determined by the formula

$$v_m' = \frac{A_c}{A_m} \frac{a_m z_x}{h_m} \cdot sec^2 \sigma \qquad \text{................(466.}$$

where z_x is the stress in the member *m* of the simple truss loaded with $P = 1$ at *D*. With these panel loads and a pole distance $p \, . \, H_x$, construct a funicular polygon for vertical loading; as before, the differences between the ordinates of this polygon and those of the *H*-curve, will give the deflections of the framed arch to a scale whose unit is $\frac{A_c \, . \, E}{H_x \;\; p}$ times that of the scale of lengths.

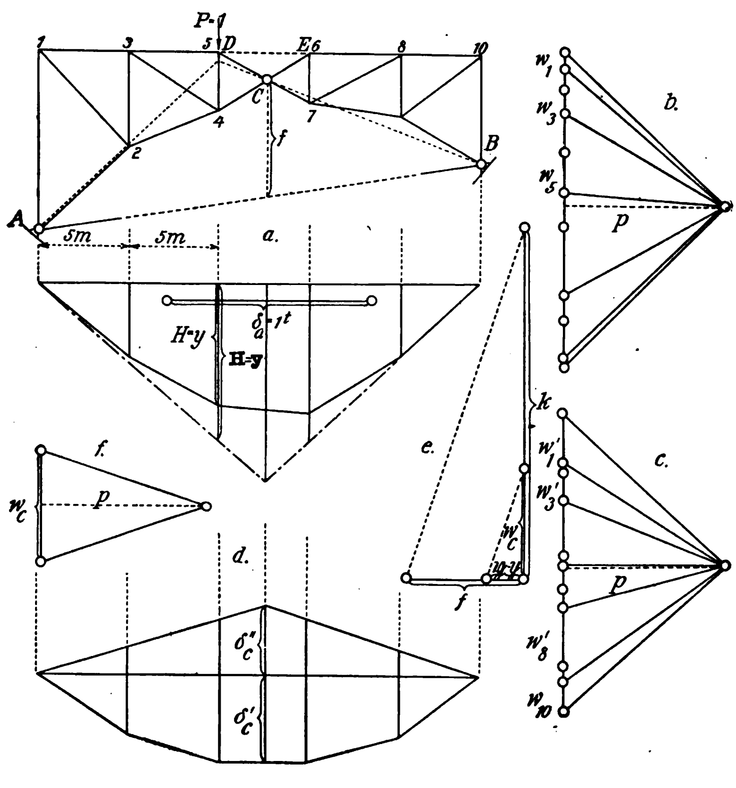
The horizontal deflections are similarly obtained, if for z_x in the above construction are used the stresses which are produced in the framework, considered as a simple truss, by a unit horizontal force applied at *D*.

If the structure has a central hinge, we again first find the elastic curve for any point D from the stresses and strains in

the three-hinged arch, assuming, however, an unyielding member ($D E$, Fig. 101) to be connected across the crown-hinge so that its angle-change will be $\omega_c = 0$. The stresses in the members will be $S = \mathbf{z_x} + \mathbf{H}_x \,.\, u$, where $\mathbf{H}_x$ is the horizontal thrust of the three-hinged arch produced by the unit load applied at D. With $\omega_c = 0$, these stresses will induce a horizontal displacement of the end A relative to B which may be calculated by the formula,

$$\overline{\delta_a} = \Sigma r S u = \Sigma r z u + \mathrm{H}_x \,.\, \Sigma r u^2$$

Fig. 101.



or, since $\Sigma r z u = -H_x \,.\, \Sigma r u^2$,

$$\overline{\delta_a} = (\mathrm{H}_x - H_x)\, \Sigma r u^2 = (\mathrm{H}_x - H_x)\, \delta_a.$$

Here H_x is the horizontal thrust of the two-hinged arch formed by introducing the rigid crown-strut, and δ_a is the corresponding horizontal deflection of A, produced by a unit

horizontal force acting at A. The quantities H_x and δ_a are to be determined by the graphic procedure described under 3.b) and 3.c).

On account of the ends being anchored, the displacement $\overline{\delta_a}$ cannot take place; there must therefore occur, at the crown, an angular rotation amounting to

$$\omega_c = \frac{\overline{\delta_a}}{f} = (\mathbf{H}_x - H_x)\frac{\delta_a}{f} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (467.$$

This will produce deflections which may be represented by the ordinates of a triangle constituting the funicular polygon for a load w_c applied at the hinge. In Fig. 101*d*, there is first drawn the deflection curve of the three-hinged arch for a unit load at D. This curve is obtained, by the method described in 3.b), as the funicular polygon of the panel loads w'; these are obtained from the strains produced by the stresses S of the three-hinged arch, using a unit $k = 20$ meters and assuming $w'_c = 0$. According to equ. (467), the load now to be assumed at the crown-hinge is

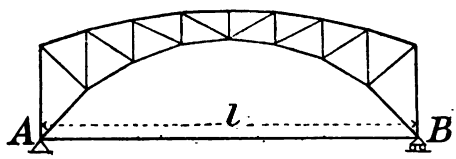
$$w_c = k.\omega_c = k.(\mathbf{H}_x - H_x)\frac{\delta_a}{f} = \frac{k}{f}(\mathbf{y}_x - y_x) \ldots\ldots (467^a.$$

where $\mathbf{y}_x$ and y_x are the ordinates of the two H-curves measured to the scale of $\delta_a = 1$. Combining the funicular triangle of the load w_c with the previously constructed deflection polygon, there will be obtained the total deflections of the three-hinged arch. (Fig. 101*d*.)

§32. The Framed Arch with Tie-Rod.

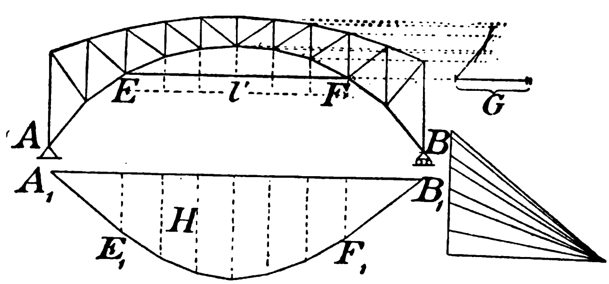
If the ends of a two-hinged arch are joined by a tension-rod to take up the horizontal thrust (Fig. 102), then, in calculating

Fig. 102.



the value of H, the effect of the stretching of this tie-rod must be considered. If A_0 is its cross-section and l its length, so that $\Delta l = \frac{H}{EA_0} l$ is its elongation, then equs. (444) and (445) will give for an arch with ends at the same level,

Fig. 103.



$$H = -\frac{\Sigma r Z u}{\Sigma r u^2 + \frac{l}{A_0}} = -\frac{\Sigma P \delta_{ma}}{\delta_{aa} + \frac{l}{EA_0}} \quad \ldots\ldots\ldots (468.$$

The H-influence line, constructed according to the preceding paragraphs, requires no modification, but the unit load determined by the displacement δ_{aa} must be increased by $1 \cdot \frac{l}{EA_0}$. Compared with the arch without a tie-rod, the value of H is reduced in the proportion of $1 : \left(1 + \frac{1}{\delta_{aa}} \cdot \frac{l}{EA_0}\right)$. A uniform temperature change in both arch and tie-rod will, in this case, produce no stresses.

If the tie-rod does not connect the end hinge-points, but joins the two panel-points $E F$ (Fig. 103) instead, then the tension produced in it will be

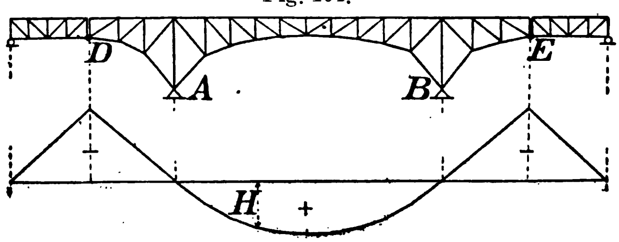
$$H = -\frac{\Sigma r Z u'}{\Sigma r u'^2 + \frac{l'}{A_0}} = -\frac{\Sigma P \delta_{me}}{\delta_e + \frac{l'}{E A_0}}$$

where u' represents the stresses in the framework produced by a unit tension along $E F$, δ_e represents the accompanying horizontal displacement of E relative to F, and δ_{me} denotes the similar horizontal displacement produced by a unit load applied at M. The latter symbol also represents the vertical deflection of M produced by a unit horizontal force in $E F$. As this tension H produces stresses and strains only in those members lying between E and F, the angle-changes of the linkwork and the panel loads w or v derived from these are also limited to the panel-points between E and F. The H-influence line obtained as the funicular polygon of the loads w will be straight in the portions $A_1 E_1$ and $F_1 B_1$ (Fig. 103).

§ 33. Other Types of Framed Arches with Hinged Abutments.

1. **The Free-Ended Cantilever Arch.** The horizontal thrust at the abutments will produce stresses and strains only in the members situated between the abutments. Accordingly, the loads w or v, employed for the construction of the H-influence line, will be limited to the panel-points between A and B and the H-curve will continue beyond these points in straight lines. (Compare the case of the Arched Rib, Fig. 74.) If the free end of the cantilever arm supports a simple truss (suspended span), so that the arch forms part (anchor span) of a "Gerber" Bridge,

Fig. 104.



then the course of the H-curve is easily specified (Fig. 104). The deflections of the panel-points of the arch, as well as those of the cantilever arm, are easily obtainable by the procedure presented in § 31 : 5.

2. **The Cantilever Arch with Ends Fixed to Horizontal Rollers.** If the ends of the cantilever arms of an arch are constrained to move in horizontal planes, there results a threefold indeterminate system. We will choose as our unknowns the horizontal thrust of the arch H, and the vertical reactions, D and E, at the end-supports. By introducing a central hinge in the arch, we may reduce the number of unknowns to two. The equations of condition are obtained by putting the horizontal displacement of A, also the vertical displacements of D and E, equal to zero.

In a free-ended cantilever arch, horizontally movable at A, let there be denoted by

$$\left.\begin{array}{ll}\delta_{ma}, & \text{the horizontal displacement of } A \text{ relative to } B\\ \delta_{md}, & \text{the vertical deflection of } D\\ \delta_{me}, & \text{the vertical deflection of } E\end{array}\right\}\begin{array}{l}\text{Produced by}\\ \text{a unit load at}\\ \text{any point } M.\end{array}$$

$\left.\begin{array}{l}\delta_{aa}\text{, the horizontal displacement of } A\\ \delta_{ad}=\delta_{da}\text{, the deflection of } D\\ \delta_{ae}=\delta_{ea}\text{, the deflection of } E\end{array}\right\}$ Produced by a unit force acting horizontally inward at A.

$\left.\begin{array}{l}\delta_{dd}\text{, the deflection of } D\\ \delta_{de}=\delta_{ed}\text{, the deflection of } E\end{array}\right\}$ Produced by a unit load at D.

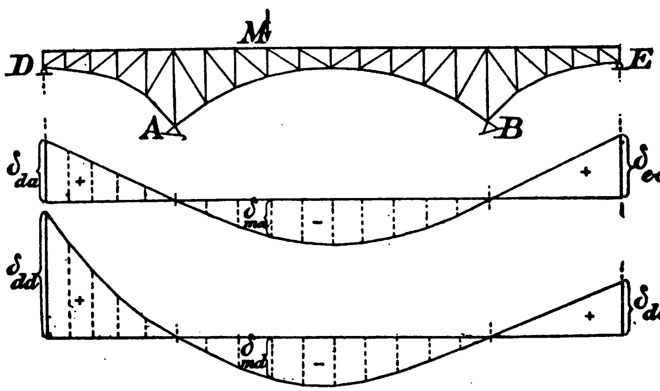
δ_{ee}, the deflection of E, produced by a unit load at E.

Then the equations of condition for H, D and E may be written,

$$\left.\begin{array}{l}\Sigma P\,.\,\delta_{ma}+H\,.\,\delta_{aa}+D\,.\,\delta_{da}+E\,.\,\delta_{ea}=0\\ \Sigma P\,.\,\delta_{md}+H\,.\,\delta_{ad}+D\,.\,\delta_{dd}+E\,.\,\delta_{ed}=0\\ \Sigma P\,.\,\delta_{me}+H\,.\,\delta_{ae}+D\,.\,\delta_{de}+E\,.\,\delta_{ee}=0\end{array}\right\}\ldots\ldots (469.$$

For the analytical treatment we will determine the series of stresses u, u_1 and u_2, which are produced in the members of the statically determinate structure when there act upon it

Fig. 105.



the forces $H=1$, $D=1$ and $E=1$, respectively. If the external loading produces stresses Z in the determinate structure, then the accession of the forces H, D and E will render

$$S=Z+H\,u+D\,u_1+E\,u_2$$

and the theorem of virtual work will therefore yield the following equations of condition, equivalent to equs. (469),

$$\left.\begin{array}{l}\Sigma\,r\,Z\,u+H\,\Sigma\,r\,u^2+D\,\Sigma\,r\,u\,u_1+E\,\Sigma\,r\,u\,u_2=0\\ \Sigma\,r\,Z\,u_1+H\,\Sigma\,r\,u\,u_1+D\,\Sigma\,r\,u_1{}^2+E\,\Sigma\,r\,u_1\,u_2=0\\ \Sigma\,r\,Z\,u_2+H\,\Sigma\,r\,u\,u_2+D\,\Sigma\,r\,u_1\,u_2+E\,\Sigma\,r\,u_2{}^2=0\end{array}\right\}\ldots (470.$$

The coefficients in these equations may be evaluated either analytically (equs. 470) or by means of a graphical treatment.

In the latter method, it is necessary to first construct the curve of deflections produced in the framework by a unit force acting along the chord AB (Fig. 105). In this curve, the ordinates under M, D and E will give the quantities δ_{ma}, δ_{da}, δ_{ea}; then, by use of the graphic method described above (§ 31:3), there is to be found the horizontal displacement δ_{aa}.

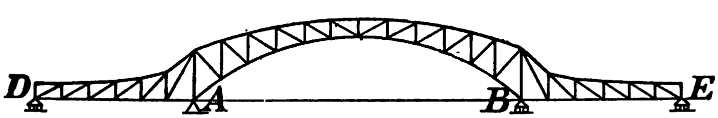
In a similar manner, construct the deflection curve of the freely supported cantilever arch for a vertical load of unity applied at D, to obtain the deflections δ_{md}, δ_{dd}, and δ_{de}. Finally, a deflection curve constructed for a unit load applied at E will give, at the points M and E, the deflections δ_{me} and δ_{ee}. If the structure is symmetrical in form, it is seen that the third deflection curve need not be drawn. There are thus obtained all the coefficients in equations (469), and solving these will give the influence values of the unknowns H, D and E. These may then be used in the equation

$$S = Z + H\,u + D\,u_1 + E\,u_2$$

to determine the influence values of the stresses in the individual members of the structure.

If the structure, instead of being anchored at the abutments A and B, is provided with a tie-rod joining these two points (Fig. 106), then the first of the equations (469), representing the horizontal displacement of A relative to B, is to be modified by replacing the zero in the second member by $-H\frac{l}{EA_0}$, where l is the length and A_0 the cross-section of the tie-rod.

Fig. 106.



If the structure has a central hinge, the problem becomes simplified since the mere omission of the end anchorages will make the structure statically determinate so that there are but two unknowns, D and E, to be determined. If the bridge is symmetrical, a single deflection curve will in this case be sufficient, namely, the deflection curve of the three-hinged arch for a unit load applied at D or E. This type of bridge is treated in greater detail in the paper cited below.*

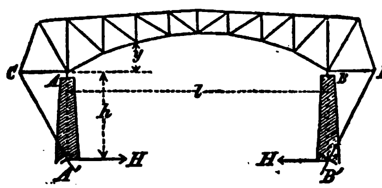
3. The Framed Arch with Anchored Ends.

This arrangement sketched in Fig. 107, proposed by Engineer A.

* *Melan,* Bogenträger mit vermindertem Horizontalschube. Österr. Monatsschr. f. d. öffentl. Baudienst, 1897.

Schnirch,* consists of an arched structure, with the ends A and B on rollers having the framework extending outward beyond these abutments and anchored by the ties CA' and DB' to the deepest practicable points of the pier-masonry. This anchorage corresponds, in effect, to a lowering of the end-hinges; the structure may be treated as an ordinary two-hinged arch with its abutments at A' and B', the portions AA' and BB' being considered as members of inelastic and unyielding material. This arrangement, by increasing the arch-rise, affords the advantage of a reduction in the horizontal thrust, especially in that due to changes of temperature; in consequence, there may be secured a marked saving of material in the masonry of the abutments, and, under certain conditions, in the superstructure as well. This anchoring will be distinctly advantageous in flat arches.

Fig. 107.



The horizontal thrust may be calculated by equ. (448). The stresses z remain the same as in the arch supported at A and B, whereas the stresses u must be determined for a horizontal unit load applied at A'. Graphically, the influence line for H may be determined as above by means of the deflection curve for a unit force at A'. Similarly, the displacements of the movable ends, A and B, may be found either by the graphic method developed above, i. e., by constructing the deflection curve for a horizontal load at A or by the analytic method of equ. (463). If any loading is placed on the structure, producing the stresses $S = Z + Hu$, the resulting combined displacement of the two rolling ends will be, by equ. (463),

$$-\Delta l = \frac{1}{E}\left[\Sigma r Z u' + H \Sigma r u u'\right] \ldots\ldots\ldots\ldots\ldots\ldots (471.$$

The stresses u', i. e., the stresses in the members producible by a unit force acting along AB, may be determined either by a force polygon or by the relation $u' = \frac{y}{y+h} \cdot u$ where y is the ordinate of the moment-center for any member measured above the arch-chord. The roller-displacement producible by temperature change is obtained by equ. (465), with $\Sigma u' s = -l$ and $\Delta l = 0$,

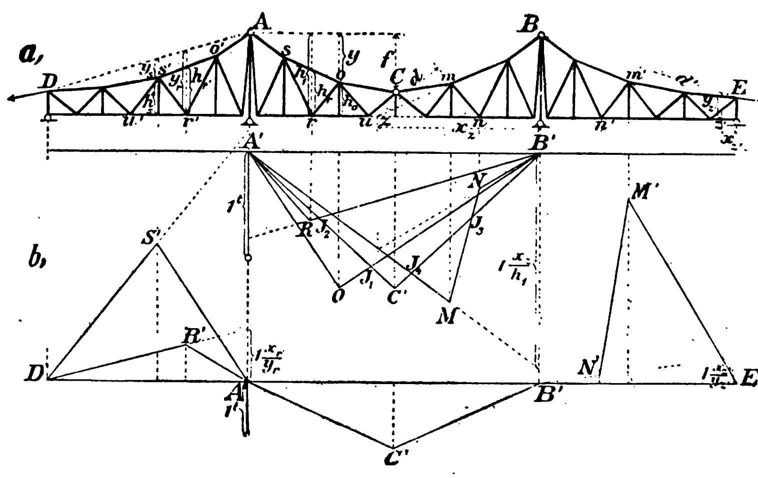
$$\Delta l_t = \omega t l \frac{\Sigma r u^2 - \Sigma r u u'}{\Sigma r u^2} = \frac{H_t}{E}\left(\Sigma r u^2 - \Sigma r u u'\right) \ldots . (472.$$

* Zeitschr. d. österr. Ing.- u. Arch.-Ver., 1884, p. 184.

§ 34. The Continuous Framed Arch and the Braced Suspension Bridge of Multiple Span.

This group of bridges corresponds to that of the arched ribs treated in § 26 and is found exemplified particularly in suspension structures. The defining condition for this type of construction is that the arches (or suspension spans) shall be hinged together at the intermediate piers and capable of horizontal displacement at these points, while the ends must be securely anchored. Such framed structures, with reference to their external reactions, are of single static indetermination.

Fig. 108.



They may be rendered determinate by introducing a central hinge in *one* of the spans.

1. We will first treat the latter case, particularly in the form of a three-span **Suspension Bridge with a Central Hinge.** The continuous tension chord is to be conceived as hinged at A and B, while the bracing is interrupted at these points of support. The method of support at these intermediate piers is to be either by means of rollers or rocker-arms; the ends of the bridge are held fixed by the roller-supports and the oblique anchorage.

a.) *Central Span.* The stresses in the main or central span are not affected by any loads in the side spans. Loads

in the central span will produce stresses according to the principles of the three-hinged arch. In Fig. 108*a*, $B'\ C'\ J_1\ O\ A'$ represents the influence line for the lower chord member $r\,u$. The factor by which this influence area is to be multiplied to give the stress in the member $r\,u$ produced by a loading of p units per unit length, is $p\cdot\frac{y}{h_0}$. With a parabolic upper chord, O would lie on the horizontal line through C'; the two triangles $A'\ C'\ B'$ and $A'\ O\ B'$ would then be of equal areas, and the positive area $A'\ O\ J_1$ would thus be equal to the negative area $B'\ C'\ J_1$. The lower chord, under this assumption, would therefore receive no stress from a uniform load covering the entire span, and its greatest stresses would be

$$L_{\max} = -L_{\min} = p.\,(Area\ A'\,O\,J_1)\,\frac{y}{h_0}.$$

In Fig. 108*a* are also drawn the influence lines for the upper chord member $o\,s$ and the web member $m\,n$. With the aid of these we may determine the limiting stresses producible in the upper chord by a uniformly distributed dead load of g per unit length and a similar live load of intensity p per unit length, by the formulae

$$U_{\max} = [\,(p+g)\ Area\ B'C'J_2 - g.Area\ J_2RA'\,]\ \frac{h_1}{h_r}.sec\,\sigma$$

$$U_{\min} = [\,g.Area\,B'\,C'\,J_2 - (p+g)\ Area J_2RA'\,]\ \frac{h_1}{h_r}\ .sec\,\sigma$$

and the corresponding stresses of the web member $m\,n$, by the formulae

$$D_{\max} = [\,(p+g)\ (Area\ B'\,N\,J_3 + Area\ J_4C'A') - g\,.\,Area\ J_3\,M\,J_4]\ \frac{h_1}{d}$$

$$D_{\min} = [\,g\ (Area\ B'\,N\,J_3 + Area\ J_4\ C'\ A') - (p+g)\ Area\ J_3M\ J_4]\,\frac{h_1}{d}.$$

Simple expressions may be written for the chord stresses in the main span. First, for any form of cable, if U_g and L_g represent the stresses due to dead load, we have

$$L_g = \left[\frac{x^2}{2}\left(\frac{2fl}{yl+2fx} - 1\right) - \frac{l^2}{8}\cdot\frac{ly-2fx}{ly+2fx}\cdot\frac{y}{f}\right]\frac{g}{h_0}$$

$$L_{\max} = L_g + \frac{px^2}{2}\left(\frac{2fl}{yl+2fx} - 1\right)\frac{1}{h_0}$$

$$L_{\min} = L_g - \frac{pl^2}{8}\cdot\frac{ly-2fx}{ly+2fx}\cdot\frac{y}{f}\cdot\frac{1}{h_0}$$

$$U_g = \left[-\frac{x^2}{2}\left(\frac{2fl}{h_1l+2fx} - 1\right) + \frac{l^2}{8}\cdot\frac{lh_1-2fx}{lh_1+2fx}\cdot\frac{h_1}{f}\right]\frac{g}{h_r}\ sec\,\sigma$$

$$U_{\max} = U_g + \frac{pl^2}{8}\cdot\frac{lh_1-2fx}{lh_1+2fx}\cdot\frac{h_1}{f}\cdot\frac{1}{h_r}\ sec\,\sigma$$

$$U_{\min} = U_g - \frac{px^2}{2}\left(\frac{2fl}{h_1l+2fx} - 1\right)\frac{1}{h_r}\ sec\,\sigma.$$

With a parabolic upper chord, the above expressions for the lower chord stresses reduce to

$$L_g = 0$$

$$L_{max} = -L_{min} = \frac{px\,(l-x)\,(l-2x)}{2\,(3l-2x)\,h_0}.$$

For the web members, the expressions for the limiting stresses become very complicated in this case; it will be more convenient to apply equ. (442), using the values of M_x and H corresponding to the critical loading.

b.) *Side Span.* The construction in the side span is to be treated as a simply supported truss subjected to the full horizontal tension arising in the main span. A load in the side span will produce the same stresses as in a framework simply resting on two supports. The upper chord will therefore receive its maximum compression when the side span is completely loaded and the main span carries no load at all; and the maximum tension is produced when the main span is completely loaded and the side span is free from load. The opposite law of loading holds true for the lower chord. If H is the horizontal tension and if **M** is the bending moment for the loading in the side span, then the stress produced by any loading of main and side span in the upper chord member $o's'$ (Fig. 108) will be

$$U = -\,(\mathbf{M} - H\,.\,y_{r'})\,\frac{sec\,\sigma}{h_{r'}} = -\left(\frac{\mathbf{M}}{y_{r'}} - H\right)\frac{y_{r'}}{h_{r'}}\,sec\,\sigma$$

and the stresses in the lower chord member $r'\,u'$ will be

$$L = (\mathbf{M} - H\,.\,y_{s'})\,\frac{1}{h_{s'}} = \left(\frac{\mathbf{M}}{y_{s'}} - H\right)\frac{y_{s'}}{h_{s'}}.$$

By these formulae, the stress influence lines may be readily constructed. These are represented in Fig. 108*b*, to one-half the scale of Fig. 108*a*, for the indicated chord members of the left-hand span; they are obtained by making the intercepts on the vertical line of the pier $= \frac{x'_{r'}}{y_{r'}}\,.\,1$ and $\frac{x'_{s'}}{y_{s'}}.1$. Here $x'_{r'}$ and $x'_{s'}$ are the horizontal distances of the panel-points r' and s' from the intermediate pier. The largest stresses producible by an assumed uniform dead load of g in the main span and of g_1 in the side spans together with a uniform live load of p per unit length will be,

$$U_{max} = [\,(p+g)\,Area\,A'\,C'\,B' - g_1\,.\,Area\,D'\,R'\,A'\,]\,\frac{y_{r'}}{h_{r'}}\,sec\,\sigma.$$

$$U_{min} = [\,g\,.\,Area\,A'\,C'\,B' - (p+g_1)\,Area\,D'\,R'\,A'\,]\,\frac{y_{r'}}{h_{r'}}\,sec\,\sigma.$$

$$L_{max} = [\,(p+g_1)\,.\,Area\,D'\,S'\,A' - g\,.\,Area\,A'\,C'\,B'\,]\,\frac{y_{s'}}{h_{s'}}.$$

$$L_{min} = [\,g_1\,.\,Area\,D'\,S'\,A' - (p+g)\,.\,Area\,A'\,C'\,B'\,]\,\frac{y_{s'}}{h_{s'}}.$$

The stresses in a web member, e. g., $m'n'$, will have the general value

$$D = -(\mathbf{M}_{z'} - H y_{z'}) \frac{1}{d'} = -\left(\frac{\mathbf{M}_{z'}}{y_{z'}} - H\right)\frac{y_{z'}}{d'},$$

where $\mathbf{M}_z'$ and y_z' refer to the point of intersection, z', of the two chord members belonging to the panel containing the diagonal. If z' falls within the span, then the same law of loading obtains for the web member as for the chords, viz., the limiting stresses arise when either the main or side span alone is completely loaded. The influence lines for this case are shown in Fig. 108*b*, in the right-hand span; the resulting stresses in the diagonal $m'n'$ will be

$$D_{max} = [(p+g)\, Area\, A'C'B' - g_1 . Area\, B'N'M'E'] \frac{y_{z'}}{d'} \cdot$$

$$D_{min} = [g . Area\, A'C'B' - (p+g_1)\, Area\, B'N'M'E'] \frac{y_{z'}}{d'} \cdot$$

The analytical expressions for the chord stresses in the side spans may be written:

$$L_g = \left[\frac{1}{2} g_1 x_s' (l_1 - x_s') - g \frac{l^2 y_s'}{8f}\right] \frac{1}{h_s'}$$

$$L_{max} = L_g + \frac{1}{2} p\, x_s' (l_1 - x_s') \frac{1}{h_s'}$$

$$L_{min} = L_g - p \frac{l^2 y_s'}{8f} \frac{1}{h_s'}$$

$$U_g = -\left[\frac{1}{2} g_1 x_r' (l_1 - x_r') - g \frac{l^2 y_r'}{8f}\right] \frac{sec\, \sigma}{h_r'}$$

$$U_{max} = U_g + p \frac{l^2 y_r'}{8f} \cdot \frac{sec\, \sigma}{h_r'}$$

$$U_{min} = U_g - \frac{1}{2} p x_r' (l_1 - x_r') \frac{sec\, \sigma}{h_r'} \cdot$$

The web stresses are to be calculated by means of the general formula (442).

2. **Continuous Braced Suspension Bridge without a Central Hinge.** If none of the spans is provided with a hinge, then the framework in each span acts as a two-hinged arch whose horizontal thrust H is conditioned by the displacements of the end-points. Let us first consider each of these arches as independent, and let

$P, P', P'' \ldots =$ loads in the 1°, 2°, 3°, …, spans.

$P\delta_{ma}, P'\delta'_{ma}, P''\delta''_{ma}, \ldots =$ the relative horizontal displacement of the ends of each span when acting as a simple truss under the above loads; considered positive if outward.

δ_{aa}, δ'_{aa}, δ''_{aa}, ... = the horizontal displacements of the ends of each span produced by unit forces acting horizontally outward at these points; accordingly

$P\,\delta_{ma} + H\,\delta_{aa}$, $P'\,\delta'_{ma} + H'\,\delta'_{aa}$, $P''\,\delta''_{ma} + H''\,\delta''_{aa}$ ... = total horizontal displacements of the ends of each span.

Then the total horizontal displacement of the anchored ends of the system must be

$$-\Delta l = P\delta_{ma} + P'\,\delta'_{ma} + P''\delta''_{ma} \ldots\ldots + H\,(\delta_{aa} + \delta'_{aa} + \delta''_{aa} \ldots\ldots)$$

whence the following value of the horizontal thrust may be written:

$$H = -\frac{P\delta_{ma} + P'\,\delta'_{ma} + P''\,\delta''_{ma} + \ldots\ldots + \Delta l}{\delta_{aa} + \delta'_{aa} + \delta''_{aa} + \ldots} \ldots\ldots (473.$$

The deflections δ are to be found either analytically or graphically according to § 31; for the analytical treatment we use the following expression corresponding to equation (444):

$$H = -\frac{\Sigma r Z u + \Sigma r Z' u' + \Sigma r Z'' u'' + \ldots + E\Delta l}{\Sigma r u^2 + \Sigma r u'^2 + \Sigma r u''^2 + \ldots\ldots} \ldots\ldots (473^a.$$

Here $r = \frac{l}{A}$ represents the ratio of length to cross-section of each member; Z, Z', Z'', ... are the stresses, produced by the actual loading, in the members of the 1°, 2°, 3°, ... spans considered as simple trusses; u, u', u'', ... are the corresponding stresses producible by a load of $H = 1$, or a force of 1, $sec\ a$ acting along the chord of each span where a is the angle of inclination.

A load in any span will thus contribute a positive horizontal force, viz., a cable-tension in the case of a suspension bridge. Having constructed the influence line for H, the influence lines for the stresses in the members are obtainable in the same manner as with the center-hinged construction.

Example. (Includes Plates II and III.) In the following we give, as an example, the complete design of G. Lindenthal's project for a railway suspension bridge over the St. Lawrence River at Quebec. The main span is 548.02 meters (= 1,800 ft.), and each of the two side spans is 207.87 m. (= 682 ft.). The bridge consists of inverted two-hinged arches swung from rocker arms and anchored at the ends. The bracing consists of vertical posts with a double set of diagonal tension members.

TABLE I.

Panel-Point	Member	l (m)	A (cm²)	$\sec\sigma$	$100\frac{l}{A}$	y (m)	h (m)	$\frac{y}{h}$	$\frac{y}{h}\sec\sigma$	$\frac{1}{h}\sec\sigma$	$v=100\frac{l}{A}\cdot\frac{y}{h^2}\sec^2\sigma$	$vy=100\frac{l}{A}\cdot\frac{y^2}{h^2}\sec^2\sigma$
	u_1	19.06	2774	1.0093	0.688				1.1861	0.0383	0.0556	0.9670
I						17.37	14.78	1.1752				
	u_2	19.15	2774	1.0134	0.692				1.2342	0.0626	0.0534	1.0653
II						22.54	17.89	1.2603				
	u_3	19.23	2838	1.0182	0.678				1.2980	0.0521	0.0454	1.1414
III						27.74	21.52	1.2889				
	u_4	19.33	2900	1.0237	0.666				1.3162	0.0437	0.0383	1.1603
IV						32.93	25.66	1.2826				
	u_5	19.47	2900	1.0300	0.672				1.3074	0.0371	0.0325	1.1536
V						38.10	30.33	1.2560				
	u_6	19.58	2900	1.0370	0.676				1.2832	0.0317	0.0275	1.1177
VI						43.28	35.51	1.2189				
	u_7	19.72	2900	1.0444	0.682				1.2864	0.0323	0.0283	1.1338
VII						36.91	29.66	1.2446				
	u_8	19.88	2964	1.0527	0.671				1.3158	0.0394	0.0347	1.1716
VIII						30.57	24.35	1.2553				
	u_9	20.06	2964	1.0616	0.677				1.3238	0.0490	0.0438	1.2034
IX						24.20	19.54	1.2387				
	u_{10}	20.23	2964	1.0711	0.682				1.2899	0.0626	0.0548	1.1461
X						17.83	15.24	1.1700				
	u_{11}	20.42	3020	1.0813	0.677				1.2651	0.0708	0.0607	1.0790
XI												
	l_1	22.48	670	1.1900	3.358				0.2085	0.0805	0.0562	0.1460
						2.19	14.78	0.1752				
	l_2	18.89	1189	1.0000	1.589				0.2179	0.0617	0.0215	0.0775
						4.66	17.89	0.2606				
	l_3	18.89	1189	1.0000	1.589				0.2748	0.0512	0.0223	0.1219
						6.22	21.52	0.2889				
	l_4	18.89	1189	1.0000	1.589				0.2858	0.0427	0.0194	0.1304
						7.25	25.66	0.2823				
	l_5	18.89	1189	1.0000	1.589				0.2693	0.0359	0.0154	0.1157
						7.77	30.33	0.2560				
	l_6	18.89	1189	1.0000	1.589				0.2374	0.0303	0.0115	0.0897
						7.77	35.51	0.2189				
	l_7	22.13	953	1.1718	2.322				0.2716	0.0362	0.0228	0.1715
						7.26	29.66	0.2446				
	l_8	22.13	953	1.1718	2.322				0.2929	0.0432	0.0298	0.2008
						6.22	24.35	0.2553				
	l_9	22.13	953	1.1718	2.322				0.2894	0.0540	0.0363	0.1979
						4.66	19.54	0.2387				
	l_{10}	22.13	953	1.1718	2.322				0.2395	0.0684	0.0380	0.1379
						2.59	15.24	0.1700				
	l_{11}	29.77	953	1.5756	3.122				0.2678	0.1033	0.0863	0.2234
												13.9491
	u_{12}	20.24	3017	1.0721	0.672				1.5861	0.0703	0.0749	1.6875
XII						22.55	15.24	1.4794				
	u_{13}	20.08	2752	1.0624	0.729				1.7404	0.0649	0.0823	2.2360
XIII						31.79	17.68	1.7970				
	u_{14}	19.92	2752	1.0535	0.724				1.9983	0.0556	0.0803	2.9115
XIV						40.75	20.42	1.9967				
	u_{15}	19.74	2730	1.0450	0.725				2.1393	0.0475	0.0736	3.3348
XV						49.80	23.77	2.0976				
	u_{16}	19.60	2730	1.0374	0.719				2.1988	0.0407	0.0643	3.4916
XVI						58.72	27.43	2.1414				
	u_{17}	19.47	2703	1.0303	0.719				2.2007	0.0349	0.0553	3.4978
XVII						67.80	31.85	2.1305				
	u_{18}	19.35	2703	1.0240	0.715				2.3529	0.0348	0.0584	3.9551
XVIII						67.6	27.43	2.4650				
	u_{19}	19.24	2643	1.0185	0.728				2.7033	0.0400	0.0786	5.3147
XIX						67.5	23.77	2.8440				
	u_{20}	19.15	2643	1.0135	0.725				3.1135	0.0461	0.1039	7.0107
XX						67.4	20.42	3.3000				

TABLE I—Continued.

Panel-Point	Member	l (m)	A (cm²)	$\sec\sigma$	$100\frac{l}{A}$	y (m)	h (m)	$\frac{y}{h}$	$\frac{y}{h}\sec\sigma$	$\frac{1}{h}\sec\sigma$	$v=100\frac{l}{A}\cdot\frac{y}{h^2}\sec^3\sigma$	$vy=100\frac{l}{A}\cdot\frac{y^2}{h^3}\sec^3\sigma$
	u_{21}	19.07	2579	1.0033	0.739				3.5853	0.0533	0.1410	9.4966
XXI						67.24	17.63	3.8045				
	u_{22}	18.99	2579	1.0061	0.737				4.1223	0.0614	0.1867	12.5456
XXII						67.1	15.24	4.3902				
	u_{23}	18.93	2579	1.0034	0.735				4.6300	0.0698	0.2348	15.7746
XXIII						67.08	13.87	4.8384				
	u_{24}	18.91	2579	1.0014	0.734				5.0463	0.0752	0.2782	18.6710
XXIV						67.04	12.80	5.2400				
	u_{25}	18.90	2579	1.0003	0.733				5.3689	0.0800	0.3146	21.1120
XXV						67.04	12.19	5.4945				
	u_{26}	18.90	2579	1.0000	0.733				5.4945	0.0820	0.3300	½22.1281 (11.0641)
XXVI												
	l_{12}	29.43	953	1.5568	3.080				0.7463	0.1023	0.2347	1.7163
						7.31	15.24	0.4794				
	l_{13}	20.96	953	1.1087	2.195				0.7076	0.0676	0.1051	1.1248
						14.19	17.68	0.7970				
	l_{14}	20.96	953	1.1087	2.195				0.9943	0.0581	0.1276	2.1977
						20.36	20.42	0.9967				
	l_{15}	20.96	1018	1.1087	2.057				1.1610	0.0504	0.1204	2.7948
						26.50	23.77	1.0976				
	l_{16}	20.96	1018	1.1087	2.057				1.2412	0.0435	0.1110	3.1851
						31.30	27.43	1.1414				
	l_{17}	20.96	1018	1.1087	2.057				1.2594	0.0376	0.0973	3.2753
						36.04	31.85	1.1305				
	l_{18}	18.90	1277	1.0000	1.480				1.2978	0.0339	0.0651	2.4811
						40.2	27.43	1.4650				
	l_{19}	18.90	1277	1.0000	1.480				1.6545	0.0393	0.0962	4.0349
						43.77	23.77	1.8440				
	l_{20}	18.90	1450	1.0000	1.303				2.0720	0.0455	0.1227	5.5613
						46.95	20.42	2.3000				
	l_{21}	18.90	1625	1.0000	1.163				2.5522	0.0528	0.1565	7.5540
						49.50	17.68	2.8045				
	l_{22}	18.90	1798	1.0000	1.051				3.0973	0.0591	0.1986	10.0545
						51.65	15.24	3.3902				
	l_{23}	18.90	1971	1.0000	0.958				3.6143	0.0689	0.2383	12.4950
						53.22	13.87	3.8384				
	l_{24}	18.90	1971	1.0000	0.958				4.0392	0.0751	0.2903	15.6135
						54.25	12.80	4.2400				
	l_{25}	18.90	2146	1.0000	0.880				4.3672	0.0800	0.3078	16.7804
						54.77	12.19	4.4945				
	l_{26}	18.90	2146	1.0000	0.880				4.4945	0.0820	0.3115	8.8858
												219.8567
1 Anchorage		48.75	7426	1.1180	0.656				1.1180			0.8192
1 Side Span												13.9491
½ Main Span												219.8567
												234.6250

1. *Influence Line for H* — By equ. (473^a),

$$H = -\frac{\Sigma r Z u + \Sigma r Z' u' + E \,.\, \Delta l}{\Sigma r u^2 + 2 \Sigma r u'^2}$$

Here Z and u, with the usual significance, refer to the members of the main span, Z' and u' refer to the members of a side span. For determining H, we will apply the approximate method (§ 31: 3. c), in which the chord strains alone are considered and the H-influence line is constructed as

the funicular polygon of the panel loads v which are obtained, according to equ. (451), from $v = r \, . \, \frac{y}{h^2} \, . \, sec^2 \sigma$. In this expression, y represents the ordinate of any panel-point measured from the line joining the points of suspension of the corresponding span; h is the vertical depth of truss; σ is the inclination from the horizontal of the chord member situated opposite the panel-point; and $r =$ the ratio, length: cross-section of the member. In the arrangement under consideration, consisting of intersecting tension-diagonals, of the two such members in each panel only that one should be considered as acting which is put in tension by the given loading. This would make the determination of the H-influence line very complicated, so that we adopt the admissible approximation of calculating each value of v from the mean of the values for the two diagonals, and considering it as applied at the mid-points of the chord-members.

We thus obtain

$$H = \frac{\mathbf{M}_v}{\Sigma \, v \, y + 2 \Sigma \, v' \, y' + 2 \frac{l}{A} \, sec^2 a} \, .$$

The extra term in the denominator takes care of the elongation in the anchor-chains (of length l, section A and inclination a). The quantities entering into the calculation are given in Table I.

In Plate II, Figs. 1—4, the H-curve is constructed as the funicular polygon of the v and v' forces and the quantities $\Sigma \, v \, y$ and $\Sigma \, v' \, y'$ are obtained as the proper intercepts of funicular polygons. In the last column of Table I, the products $v \, y$ are also computed and their sum is found to be

$$100 \left(\frac{1}{2} \Sigma \, v \, y + \Sigma \, v' \, y' + \frac{l}{A} \, sec^2 a \right) = 234.62.$$

The pole distance of the v and v' force polygons having been taken as $p = 3.875$ times the unit to which the 100-fold v's were plotted, therefore in the expression $H = \frac{\xi}{N} \, . \, G$, where ξ is any ordinate of the H-curve, the value of N must be $N = \frac{2 \times 234.62}{3.875} = 121.1$ meters, which agrees with the construction. The ordinates of the H-curve are compiled in Table II.

TABLE II.

Panel-Point	0	I	II	III	IV	V	VI	VII	VIII	IX	X	XI			Σ
ξ	0	1.95	3.20	3.90	4.42	4.79	4.79	4.42	3.96	3.35	2.13	0			36.91

Panel-Point	XII	XIII	XIV	XV	XVI	XVII	XVIII	XIX	XX	XXI	XXII	XXIII	XXIV	XXV	Σ
ξ	22.10	41.76	60.66	78.18	95.21	111.10	126.55	141.18	154.31	166.53	177.32	185.86	191.47	194.46	1746.68

Hence, if P is the load per panel-point, for a side span completely loaded,

$$H = \frac{36.91}{121.1} P = 0.3043P;$$

for the main span completely loaded,

$$H = 2 \cdot \frac{1746.69}{121.1} P = 28.798 \ P.$$

The horizontal thrust produced by temperature is

$$H_t = \frac{E \omega t \ (L + 2 L_1 sec^2 a)}{\Sigma r u^2};$$

here $L = 548.0$ m., $L_1 = 207.87$ m., $sec^2 \ a = 1.0752$, $E = 2040$ tonnes/cm.², $\omega t = \frac{40}{80,000}$, $E \omega t = 1.02$ tonnes/cm.², so that

$$H_t = \frac{1.02 \times 994.2}{2 \times 234.62} = 216.16 \text{ tonnes.}$$

2. *Chord Stresses.* The panel-points are numbered consecutively from 0 to X1 in the side span and from XI to XXV to the mid-point of the center span. The lower chord members will be denoted by L, the upper chord members by U, the diagonals inclined downward to the right by D, those slanting downward to the left by D', and the posts by V. As an index to each member will be attached the number of the corresponding panel (1-11 in the side span and 12-26 in the center span), the posts being designated by the number of the panel-point. In Plate II, Fig. 4, are drawn the influence lines for the chord stresses in the main span. Since the upper chord conforms to a parabola, all its panel-points will have the same value of $\frac{\mathbf{M}}{y}$; the lower chord influence lines are therefore easily constructed, all of them being represented by triangles of the same altitude, viz.

$$\frac{1}{4} \frac{L}{f} \cdot G = \frac{1}{4} \frac{548.0}{54.8} \times 121.1 = 302.75 \text{ m.}$$

The altitudes of the influence triangles for the upper chord stresses are given by $\frac{\mathbf{M}}{y_1} = \frac{x \ (L - x)}{L \cdot y_1} G$, where x and y_1 are the coordinates of a lower chord panel-point referred to the panel-point XI. We thus obtain the following values:

There were thus constructed the upper chord influence lines in Plate II, Fig. 4.

Panel-Point	XI	XII	XIII	XIV	XV	XVI	XVII	XVIII	XIX	XX	XXI	XXII	XXIII	XXIV	XXV
$m = \frac{\mathbf{M}}{y_1}$	0	98.14	134.35	151.23	158.49	161.46	16?.73	180.03	196.69	211.14	223.30	233.07	240.31	245.18	247.55

The limiting stresses in the members produced by a uniform load (p) will now be found from the areas of the influence lines.

With a as the panel-length, let

Φ = area of H-curve in the main span = 3493.38

Φ_1 = area of H-curve in the side span = $36.91a$

F_1 = the negative portion of the influence area for a chord member in the main span,

F = area of the $\frac{M}{y}$ triangle = $\frac{M}{y} \cdot \frac{29a}{2}$.

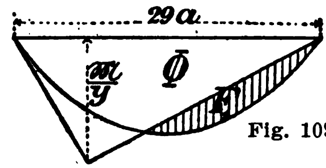


Fig. 109

Then,

$$F_{max} = F + F_1 - \Phi, \qquad F_{min} = F_1 + 2\Phi_1,$$

$$S_{max} = F_{max} \cdot \frac{y}{h} \cdot sec\,\sigma \cdot \frac{p}{G}, \qquad S_{min} = -F_{min} \cdot \frac{y}{h} \cdot sec\,\sigma \cdot \frac{p}{G}.$$

For all the lower chord members, $F = 302.75 \times 14.5.a = 4389.88.a$ (constant). For any member and any loading, that influence line should be used which refers to the panel-point belonging to the positively stressed diagonal in the panel.

In Table III are calculated the chord stresses produced by the moving load. In the actual design, the moving load was $p = 4464$ kilograms per meter (= 3000 pounds per linear foot), or a panel load of $pa = 4464 \times 18.9 = 84{,}370$ kg. We thus have $\frac{p\,a}{G} = \frac{84{,}370}{121.1} = 696.7$, so that the quantities $\frac{1}{a} F_{max}$ and $\frac{1}{a} F_{min}$ in the table must be multiplied by the factor $0.6967 \cdot \frac{y}{h} \cdot sec\,\sigma$ in order to obtain the chord stresses in tonnes.

It was not necessary to draw the chord influence lines for the side spans, since just two different cases of loading need be considered in order to obtain the extreme stresses in all the members: one of the side spans completely loaded, or the main span and the other side span covered with load. The data for computation are arranged in Table IV.

Remark on Tables III and IV: The oblique lines in columns 6 and 10, and 7 and 11, respectively, indicate which panel-point is to be the center of moments for each chord stress according to the diagonal considered acting.

With reference to the dead load, it is assumed that this is carried directly by the tension-chord acting as a cable and consequently the bracing receives no stress from this source. This condition may be realized by making the diagonals adjustable in length and leaving them loose during erection, so that only the posts serve to transmit the load to the cable. Not until the entire construction, including the roadway, is completed, are the diagonals to be adjusted; they should be given a slight initial tension at a mean temperature of about 10°C. (= 50°F.). An approximately uniform distribution of the dead load corresponds to the parabolic curve adopted for the tension-chord and to the ratio of rises or versines in the main and side spans of

$$f : f_1 = L^2 : L_1^2.$$

The stresses in the cable produced by the dead load of 8900 kg. per meter (= 6,000 lbs. p.l.f.) are computed by the formula

$$U_g = \frac{1}{8} \cdot 8900 \cdot \frac{548^2}{54.8} sec\,\sigma = 6096.5 \cdot sec\,\sigma \text{ (tonnes)}.$$

These values are given in Table V.

TABLE III.

Maximum Moving-Load Chord Stresses in the Main Span.

Panel-Point	Chord Member	$F\frac{1}{a}$ (m.)	$F_1\frac{1}{a}$ (m)	$F_{max}\frac{1}{a}$ (m)	Diagonals Acting	$\frac{y}{h}\sec\sigma$	S_{max} (t.)	$F_{min}\frac{1}{a}$ (m)	Diagonals Acting	$\frac{y}{h}\sec\sigma$	S_{min} (t)
	L_{12}					0.7463	+805.9			0.7463	—378.0
XII		4389.9	654.5	1551.0	D_{12}			728.3	D'_{12}		
	L_{13}					0.5315	573.9			0.8836	—411.1
XIII		4389.9	595.2	1491.7	$D_{13}D_{14}$			669.0	$D'_{13}D'_{14}$		
	L_{14}					0.8836	917.7			1.1050	—488.3
XIV		4389.9	536.7	1433.2	$D_{14}D_{15}$			610.5	$D'_{14}D_{15}$		
	L_{15}					1.1050	1102.7			1.1050	—488.3
XV		4389.9	477.1	1373.6	$D_{15}D'_{16}$			550.9	$D_{15}D_{16}$		
	L_{16}					1.2655	1157.4			1.2169	—466.3
XVI		4389.9	417.0	1313.5	$D'_{16}D'_{17}$			490.8	$D_{16}D_{17}$		
	L_{17}					1.2534	1091.7			1.2655	—431.9
XVII		4389.9	354.2	1250.7	$D'_{17}D_{18}$			428.0	$D_{17}D'_{18}$		
	L_{18}					1.1305	984.6			1.4650	—374.0
XVIII		4389.9	293.3	1189.8	$D_{18}D_{19}$			367.1	$D'_{18}D'_{19}$		
	L_{19}					1.4650	1213.9			1.8440	—391.9
XIX		4389.9	231.8	1128.3	$D_{19}D_{20}$			305.6	$D'_{19}D_{20}$		
	L_{20}					1.8440	1449.0			2.3000	—395.8
XX		4389.9	173.6	1070.1	$D_{20}D_{21}$			247.4	$D'_{20}D_{21}$		
	L_{21}					2.3000	1714.3			2.3000	—395.8
XXI		4389.9	117.4	1013.9	$D_{21}D_{22}$			191.2	$D_{21}D_{22}$		
	L_{22}					2.8045	1980.8			2.8045	—373.0
XXII		4389.9	69.4	965.9	$D_{22}D_{23}$			143.2	$D_{22}D_{23}$		
	L_{23}					3.3902	2281.3			3.3902	—337.8
XXIII		4389.9	33.9	930.4	$D_{23}D_{24}$			107.7	$D_{23}D_{24}$		
	L_{24}					3.8384	2488.0			3.8384	—287.5
XXIV		4389.9	11.6	909.1	$D_{24}D_{25}$			85.4	$D_{24}D_{25}$		
	L_{25}					4.2400	2682.7			4.2400	—252.0
XXV		4389.9	2.2	898.7	$D_{25}D'_{26}$			76.0	$D_{25}D_{26}$		
	L_{26}					4.4945	2807.4			4.4945	—237.6
	U_{12}					1.5861	—250.5			1.5861	+2615.2
XII		1423.1	2297.0	226.7	D_{12}			2370.8	D_{12}		
	U_{13}					1.9091	—369.6			1.5717	2579.0
XIII		1948.1	1823.7	278.4	$D_{13}D_{14}$			1897.5	$D'_{13}D_{14}$		
	U_{14}					2.1035	—453.9			2.1035	2464.6
XIV		2192.8	1610.9	310.3	$D_{14}D'_{15}$			1684.7	$D_{14}D_{15}$		
	U_{15}					2.0866	—450.2			2.1920	2381.9
XV		2298.1	1488.6	293.3	$D_{15}D'_{16}$			1562.4	$D_{15}D_{16}$		
	U_{16}					2.1761	—442.9			2.2215	2280.2
XVI		2341.2	1402.1	249.9	$D'_{16}D'_{17}$			1475.9	$D_{16}D_{17}$		
	U_{17}					2.2063	—383.5			2.1951	2168.0
XVII		2330.6	1346.3	183.5	$D'_{17}D_{18}$			1420.1	$D_{17}D_{18}$		
	U_{18}					2.5242	—387.5			2.5242	2067.1
XVIII		2610.5	1103.7	220.8	$D_{18}D_{19}$			1177.5	$D_{18}D_{19}$		
	U_{19}					2.8966	—502.4			2.8966	1943.2
XIX		2852.0	890.8	249.4	$D_{19}D_{20}$			964.6	$D_{19}D_{20}$		
	U_{20}					3.3445	—626.6			3.3445	1802.6
XX		3061.6	701.2	269.4	$D_{20}D_{21}$			775.0	$D_{20}D_{21}$		
	U_{21}					3.8399	—730.6			3.8399	1610.6
XXI		3237.7	529.3	273.6	$D_{21}D_{22}$			603.1	$D_{21}D_{22}$		
	U_{22}					4.4170	—829.6			4.4170	1406.1
XXII		3369.5	383.9	270.0	$D_{22}D_{23}$			457.7	$D_{22}D_{23}$		
	U_{23}					4.8548	—869.3			4.8548	1148.3
XXIII		3484.5	266.3	257.4	$D_{23}D'_{24}$			340.1	$D_{23}D_{24}$		
	U_{24}					5.2473	—872.4			5.2473	916.0
XXIV		3555.2	177.2	239.0	$D_{24}D'_{25}$			251.0	$D_{24}D_{25}$		
	U_{25}					5.2416	—871.4			5.4961	759.8
XXV		3589.5	124.6	220.7	$D_{25}D'_{26}$			198.4	$D_{25}D_{26}$		
	U_{26}					5.4945	—843.6			5.4945	759.6

TABLE IV.

Maximum Moving-Load Chord Stresses in the Side Span.

Panel-Point	Chord Member	y (m)	M (t.m.)	$H_1.y$ (t.m.)	$M-H_1.y$ (t.m.)	Diagonals Acting	$\frac{1}{h}\sec\sigma$	S_{min} (t.)	$(H+H_1)y$	Diagonals Acting	$\frac{1}{h}\sec\sigma$	S_{max} (t.)
		Left Span Completely Loaded.							Middle and Right Spans Completely Loaded.			
	L_1						0.08051	+636.4			0.08051	—512.2
I		2.59	7971.6	66.8	7904.8	D_2			6361.2	D'_2		
	L_2						0.06765	534.8			0.05591	—640.1
II		4.66	14349.0	119.7	14229.3	D_2D_3			11450.0	$D'_2D'_3$		
	L_3						0.05591	794.5			0.04646	—709.2
III		6.22	19132.0	159.7	18972.3	$D_3D'_4$			15266.7	D'_3D_4		
	L_4						0.03894	862.0			0.04646	—709.2
IV		7.25	22320.7	186.2	22134.5	$D'_4D'_5$			17811.1	D_4D_5		
	L_5						0.03297	782 0			0.03894	—693.6
V		7.77	23915.0	199 5	23715.5	$D'_5D'_6$			19083.3	D_5D_6		
	L_6						0.02815	667.6			0.03297	—629.2
VI		7.77	23915.0	199.5	23715.5	D'_6D_7			19083.3	$D_6D'_7$		
	L_7						0.03297	782.0			0.03953	—704.2
VII		7.25	22320.7	186.2	22134.5	D_7D_8			17811.1	$D'_7D'_8$		
	L_8						0.03953	875.1			0.04813	—734.8
VIII		6.22	19132.0	159.7	18972.3	$D_8D'_9$			15266.7	D'_8D_9		
	L_9						0.05997	853.4			0.04813	—734.8
IX		4.66	14349.0	119.7	14229.3	$D'_9D'_{10}$			11450.0	D_9D_{10}		
	L_{10}						0.07690	607.9			0.05997	—686.7
X		2.59	7971.6	66.8	7904.8	D'_{10}			6361.2	D_{10}		
	L_{11}						0.10338	817.2			0.10338	—657.6
	U_1						0.06827	—513.8			0.06827	+2912.5
I		17.37	7971.6	446.0	7525.6	D_2			42657 0	D'_2		
	U_2						0.05666	—780.2			0.06857	2925.1
II		22.55	14349.0	579.0	13770.0	D_2D_3			55379.2	$D'_2D'_3$		
	U_3						0.04731	—871.5			0.05692	3152.4
III		27.73	19132.0	712.1	18419.9	$D_3D'_4$			68101.5	D'_3D_4		
	U_4						0.04757	—876.3			0.03986	3221.9
IV		32.93	22320.7	845.1	21575.6	$D'_4D'_5$			80823.7	D_4D_5		
	U_5						0.04013	—861.7			0.03396	3176.6
V		38.10	23915.0	978.1	22936.9	$D'_5D'_6$			93545.8	D_5D_6		
	U_6						0.03419	—784.1			0.02920	3103.1
VI		43.28	23915.0	1111.1	22803.9	D'_6D_7			106268.0	$D_6D'_7$		
	U_7						0.03520	—752.4			0.02940	3124.0
VII		36.91	22320.7	947.6	21373.1	D_7D_8			90627.3	$D'_7D'_8$		
	U_8						0.04324	—793.4			0.03550	3217.2
VIII		30.57	19132.0	784.9	18347.1	$D_8D'_9$			75061.3	D'_8D_9		
	U_9						0.04360	—800.0			0.05433	3228.5
IX		24.20	14349.0	621.3	13727.7	$D'_9D'_{10}$			59420.3	D_9D_{10}		
	U_{10}						0.05482	—752.6			0.07828	3076.7
X		17.83	7971.6	457.7	7513.9	D'_{10}			43779 5	D_{10}		
	U_{11}						0.07097	—533.2			0.07097	3106.9

The *temperature changes,* which give rise to the horizontal force computed above, produce stresses in the chords of the structure amounting to $H_t \cdot \frac{y}{h} \cdot \sec\sigma$. For a range of temperature of $\pm 40°$C. ($= 72°$F.), we found above that $H_t = \pm 216.16$ tonnes; the resulting chord stresses are listed in Table V. This Table V contains in addition, in the last two columns, the extreme values of the chord stresses producible by the combined action of dead load, live load and temperature variation.

TABLE V.

Summary of Maximum Chord Stresses.

Chord Member	Dead Load Stress (t)	Live Load Stress Max. (t)	Live Load Stress Min. (t)	Temperature Effect −40°C (t)	Diagonal Acting	+40°C. (t)	Diagonal Acting	Total Stresses Max. (t)	Total Stresses Min. (t)
U_1	+6153.1	+2912.5	—513.8	+256.4		— 256.4		+9322.0	+5382.9
U_2	6178.2	2925.1	—780.2	257.4	D'_2	— 276.2	D_2	+9360.7	+5121.8
U_3	6207.5	3152.4	—871.5	277.6	D'_3	— 282.4	D_3	+9637.5	+5053.6
U_4	6241.0	3221.9	—876.3	283.6	D_4	— 283.9	D'_4	+9746.5	+5080.8
U_5	6279.4	3176.6	—861.7	279.7	D_5	— 285.6	D'_5	+9735.7	+5132.1
U_6	6322.1	3103.1	—784.1	273.2	D_6	— 281.5	D'_6	+9698.4	+5256.5
U_7	6366.7	3124.0	—752.4	275.0	D'_7	— 281.0	D_7	+9765.7	+5333.3
U_8	6417.3	3217.2	—793.4	283.2	D'_8	— 285.7	D_8	+9917.7	+5496.2
U_9	6471.6	3228.5	—800.0	284.2	D_9	— 288.1	D'_9	+9984.3	+5383.5
U_{10}	6529.5	3076.7	—752.6	270.9	D_{10}	— 286.8	D'_{10}	+9877.1	+5490.1
U_{11}	6591.7	3103.9	—533.2	273.5		— 273.5		+9972.2	+5784.0
L_1		—512.2	+636.4	— 45.2	Same as above	+ 45.2	Same as above	— 557.4	+ 681.6
L_2		—640.1	534.8	— 56.3		37.9		— 696.4	+ 572.7
L_3		—709.2	794.5	— 62.5		56.3		— 771.7	+ 850.8
L_4		—700.2	862.0	— 62.4		61.1		— 771.6	+ 923.1
L_5		—693.6	782.0	— 61.1		55.3		— 754.7	+ 837.3
L_6		—629.2	667.6	— 55.3		47.3		— 684.5	+ 714.9
L_7		—704.2	782.0	— 62.0		55.4		— 766.2	+ 837.4
L_8		—734.8	875.1	— 64.7		62.0		— 799.5	+ 937.1
L_9		—734.8	853.4	— 64.7		60.5		— 799.5	+ 913.9
L_{10}		—686.7	607.9	— 60.5		43.0		— 747.2	+ 650.9
L_{11}		—657.6	817.2	— 57.9		57.9		— 715.5	+ 815.1
U_{12}	+6536.1	+2615.2	—250.5	+342.9		— 342.9		+9494.2	+5942.7
U_{13}	6476.9	2579.0	—396.6	339.7	D_{13}	— 412.7	D'_{13}	+9395.6	+5694.6
U_{14}	6422.7	2464.6	—453.9	409.2	D_{14}	— 454.7	D'_{14}	+9296.5	+5514.1
U_{15}	6370.8	2381.9	—450.2	451.1	D_{15}	— 473.8	D'_{15}	+9203.8	+5446.8
U_{16}	6324.5	2280.2	—442.9	470.4	D_{16}	— 480.2	D'_{16}	+9075.1	+5401.4
U_{17}	6281.2	2168.0	—383.5	474.5	D'_{17}	— 476.9	D_{17}	+8943.7	+5420.8
U_{18}	6242.8	2067.1	—387.5	471.6	D_{18}	— 545.6	D'_{18}	+8781.5	+5309.7
U_{19}	6200.3	1943.2	—502.4	542.7	D_{19}	— 626.1	D'_{19}	+8695.2	+5080.8
U_{20}	6178.8	1802.6	—626.6	623.1	D_{20}	— 723.0	D'_{20}	+8604.5	+4829.2
U_{21}	6153.2	1610.6	—730.6	720.0	D_{21}	— 830.0	D'_{21}	+8483.8	+4592.6
U_{22}	6133.7	1406.1	—829.6	827.4	D_{22}	— 954.8	D'_{22}	+8367.2	+4349.3
U_{23}	6117.2	1118.3	—869.3	952.2	D_{23}	—1049.3	D'_{23}	+8217.7	+4258.6
U_{24}	6105.0	916.0	—872.4	1047.4	D_{24}	—1134.3	D'_{24}	+8068.4	+4098.3
U_{25}	6098.3	759.8	—871.4	1133.0	D_{25}	—1188.0	D'_{25}	+7991.1	+4038.9
U_{26}	6096.5	759.6	—843.6	1187.8	D_{26}	—1187.8	D'_{26}	+8043.9	+4065.1
L_{12}		—378.0	+805.9	—161.3	Same as above	+ 161.3	Same as above	— 539.3	+ 967.2
L_{13}		—411.1	573.9	—191.0		114.9		— 602.1	+ 688.8
L_{14}		—488.3	917.7	—238.9		191.0		— 727.2	+1188.7
L_{15}		—488.3	1102.7	—263.0		238.9		— 751.3	+1341.6
L_{16}		—466.3	1157.4	—273.6		263.0		— 739.9	+1420.4
L_{17}		—431.9	1091.7	—273.6		270.9		— 705.5	+1362.6
L_{18}		—374.0	984.6	—316.7		244.4		— 690.7	+1229.0
L_{19}		—391.9	1213.9	—398.6		316.7		— 790.5	+1530.6
L_{20}		—395.8	1449.0	—497.2		398.6		— 893.0	+1847.6
L_{21}		—395.8	1714.3	—606.2		497.2		—1002.0	+2211.5
L_{22}		—373.0	1980.8	—732.8		606.2		—1005.8	+2587.0
L_{23}		—337.8	2281.3	—829.7		732.8		—1167.5	+3014.1
L_{24}		—237.5	2488.0	—916.5		829.7		—1204.0	+3317.7
L_{25}		—252.0	2682.7	—971.6		916.5		—1223.6	+3599.2
L_{26}		—237.6	2807.4	—971.6		971.6		—1209.2	+3779.0

3. *Web Stresses.* Employing the notation indicated in the adjoining diagram, Fig. 110, the stress in any diagonal in the main span will be

$$D = \left(Z - H\frac{y}{z}\right) = \frac{y}{z}\left(\frac{Z \cdot z}{y} - H\right).$$

Here **Z** is the stress for a simply supported truss, and its influence line

may be constructed in the well known manner. Using an intercept $= G . \frac{x}{y}$, laid off on the left reaction vertical, we obtain the influence line for $\frac{Z z}{y} = \frac{M}{y}$; this is represented by $A_1 m_1 n_1 B_1$ (Fig. 111), from which the ordinates of the H-curve are to be subtracted. The maximum tension in the diagonal, D, is represented by the intercepted area (shaded) F_{max}, and its value will be

$$D_{max} = \frac{p}{G} \cdot \frac{y}{z} \cdot F_{max}.$$

The maximum tension in the opposite diagonal D' is similarly determined

Fig. 110.

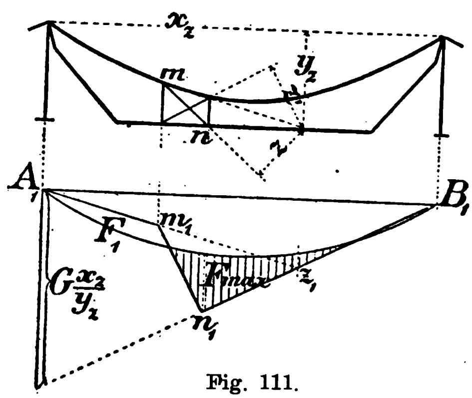


Fig. 111.

by the negative portion (F_1) of the influence area augmented by twice the area (Φ) of the H-curve in the side span so that

$$D'_{max} = \frac{p}{G} \cdot \frac{y}{z'} \, (F_1 + 2\phi)$$

In Plate III, Fig. 1, are constructed the influence lines for the diagonals, and the data for their computation are arranged in Table VI. The ordinates of the H-curve are those of Plate II reduced to $\frac{1}{4}$ the scale, so that the unit load $G = \frac{1}{4} \times 121.1 = 30.3$ meters. Column 4 of Table VI gives the intercepts $\left(G \frac{x}{y}\right)$ used in constructing the influence lines; and the areas in column 5, when multiplied by the factor $\frac{p\,a}{G} \cdot \frac{y}{z} = \frac{84.37}{30.3} \frac{y}{z} = 2.786 \frac{y}{z}$, give the maximum tensions in the diagonals producible by the moving load. The temperature stresses, with $H_t = 216.16$ t., are given by $D_t = 216.16 \frac{y}{z}$; these are listed in column 9 in the table.

TABLE VI.
Stresses in the Diagonals of the Main Span.

Member	x (m)	y (m)	$\frac{x}{y} \cdot G$ ($G=30.3$) (m)	$\frac{1}{a} F_{max}$ (m)	z (m)	$\frac{y}{z}$	L. L. D_{max} (t)	Tem. D_t (t)	L. L. + Tem. (t)
D_{13}	−106.52	−37.79	85.47	371.22	−72.23	0.5232	540.3	113.1	653.4
D_{14}	−79.85	−24.75	97.85	391.98	−73.30	0.3376	368.1	73.0	441.1
D_{15}	−57.36	−14.05	123.79	534.03	−77.42	0.1815	269.6	39.2	308.8
D_{16}	−39.62	−5.52	217.80	1371.81	−83.82	0.0658	251.1	14.2	265.3
D_{17}	−24.32	+1.95	318.17	$-\left\{\begin{matrix}4262.99\\18.45\end{matrix}\right.$	−92.05	0.0212	252.4	4.6	257.0
D_{18}	+251.15	+66.69	114.20	259.45	+102.26	0.6522	470.7	141.0	611.7
D_{19}	269.59	66.75	122.48	254.19	+97.23	0.6865	485.4	148.4	633.8
D_{20}	288.64	66.75	131.13	252.29	92.96	0.7180	503.9	155.2	659.1
D_{21}	312.71	66.75	142.07	264.36	90.52	0.7374	542.3	143.1	685.4
D_{22}	345.02	66.63	157.03	297.78	93.94	0.7092	587.5	153.3	740.8
D_{23}	391.23	66.63	177.74	351.65	104.39	0.6394	625.5	138.2	763.7
D_{24}	481.87	66.57	219.83	486.25	139.11	0.4779	646.4	103.3	749.7
D_{25}	767.46	66.44	350.27	936.85	281.72	0.2358	614.6	51.0	665.6
D_{26}	$=3.5\,p\,a.\sec\beta=3.5\times84.37\times1.838=$						540.6	0	540.6
D'_{13}		−37.79	$\left.\begin{matrix}F_1=147.26\\2\Phi_1=18.45\end{matrix}\right\}$	= 165.71	−107.35	0.3527	162.6	76.3	238.9
D'_{14}		−24.75	$\left.\begin{matrix}162.26\\18.45\end{matrix}\right\}$	= 180.71	−109.72	0.2256	113.4	48.8	162.2
D'_{15}		−14.05	$\left.\begin{matrix}293.81\\18.45\end{matrix}\right\}$	= 312.26	−114.30	0.1229	106.8	26.6	133.4
D'_{16}		−5.52	$\left.\begin{matrix}1100.43\\18.45\end{matrix}\right\}$	= 1118.86	−120.70	0.0457	142.2	9.9	152.1
D'_{17}		1.95		− 4404.21	−128.16	−0.0152	186.5	3.3	189.8
D'_{18}		66.69	$\left.\begin{matrix}38.56\\18.45\end{matrix}\right\}$	= 57.01	113.08	0.5898	93.5	127.5	221.0
D'_{19}		66.75	$\left.\begin{matrix}27.89\\18.45\end{matrix}\right\}$	= 46.34	106.52	0.6266	80.8	135.4	216.2
D'_{20}		66.75	$\left.\begin{matrix}27.57\\18.45\end{matrix}\right\}$	= 46.02	100.31	0.6655	85.2	143.9	229.1
D'_{21}		66.75	$\left.\begin{matrix}41.05\\18.45\end{matrix}\right\}$	= 59.50	96.86	0.6891	114.1	149.0	263.1
D'_{22}		66.63	$\left.\begin{matrix}69.49\\18.45\end{matrix}\right\}$	= 87.94	98.63	0.6755	164.4	146.0	280.4
D'_{23}		66.63	$\left.\begin{matrix}122.60\\18.45\end{matrix}\right\}$	= 141.05	107.90	0.6186	242.7	133.7	375.4
D'_{24}		66.57	$\left.\begin{matrix}245.60\\18.45\end{matrix}\right\}$	= 264.05	141.73	0.4690	344.5	101.4	445.9
D'_{25}		66.44	$\left.\begin{matrix}676.40\\18.45\end{matrix}\right\}$	= 694.85	282.85	0.2349	454.1	50.8	504.9
D'_{26}		as for D_{26}					540.6	0	540.6

To find the stresses in the *posts of the main span*, the same method of treatment will be followed. If z is the distance of the mth post from the intersection of the mth upper chord member with the $(m+1)$th lower chord member, and if y is the ordinate of this intersection point, then

$$V_{max} = -\frac{p}{G} \cdot \frac{y}{z} \cdot F_{max}.$$

F_{max} is the positive portion of the influence line for the post constructed by using the intercept $G\frac{x}{y}$ on the reaction vertical. With the exception of V_{17}, the influence lines for the posts are easily obtained from those of the diagonals; they are drawn in Plate III, Fig. 1, as dash and dot lines. The minimum stress in the posts occurs when the system of diagonals D' is acting; for this case,

$$V_{min} = -\frac{p}{G}\frac{y'}{z'}(F_1 + 2\Phi_1)$$

where z' and y' refer to the point of intersection of the $(m+1)$th upper chord, with the mth lower chord member, and F' represents the negative portion of the corresponding influence area. The quantities for computa-

tion are arranged in Table VII; it should be observed that the stresses in columns 10 and 11 are obtained by multiplying the areas in columns 2 and 6 by the factors 2.786 $\frac{y}{z}$ and 2.786 $\frac{y'}{z'}$ respectively.

The stresses V_{max} are found to be larger throughout than V_{min}; only the former need therefore be considered, and it consequently suffices to calculate the temperature stresses for rising temperature only (which brings the diagonals D into action); these temperature stresses are given by the expression $216.16 \frac{y}{z}$.

TABLE VII.
Stresses in the Posts in the Main Span.

Member	$\frac{1}{a} F_{max}$ (m)	y (m)	z (m)	$\frac{y}{z}$	$\frac{1}{a}(F_1+2\Phi_1)$ (m)	y' (m)	z' (m)	$\frac{y'}{z'}$	Moving Load V_{max} (t)	Moving Load V_{min} (t)	Temp. V_t (t)	$V_{max}+V_t$ (t)
V_{12}	310.73	56.39	183.48	0.3073					—265.6		— 66.4	—332.0
V_{13}	296.99	37.79	144.32	0.2619	103.22 / 18.45	24.75	117.65	0.2104	—216.4	— 71.2	— 56.6	—273.0
V_{14}	317.35	24.75	136.54	0.1812	144.33 / 18.45	14.05	114.05	0.1232	—161.1	— 55.8	— 39.2	—200.3
V_{15}	446.26	14.05	132.95	0.1057	640.69 / 18.45	5.52	115.21	0.0426	—131.2	— 78.1	— 22.9	—154.1
V_{16}	1216.17	5.52	134.11	0.4114	2892.44	1.95	118.81	0.01642	—139.2	—132.1	— 8.9	—148.1
V_{17}	227.92	66.69	124.05	0.5376	928.23 / 18.45	8.53	123.99	0.0688	—340.8	—177.7	—125.9	—466.7
V_{18}	205.43	66.69	118.87	0.5610	26.67 / 18.45	66.75	137.31	0.4861	—320.6	— 61.0	—121.2	—441.8
V_{19}	200.89	66.75	118.41	0.5638	24.67 / 18.45	66.75	137.46	0.4856	—315.1	— 58.2	—121.9	—437.0
V_{20}	200.50	66.75	118.56	0.5630	35.33 / 18.45	66.75	142.64	0.4680	—314.0	— 70.0	—121.7	—435.7
V_{21}	193.65	66.75	123.74	0.5394	59.27 / 18.45	66.63	156.05	0.4269	—290.6	— 92.3	—116.6	—407.2
V_{22}	245.14	66.63	137.16	0.4858	105.90 / 18.45	66.63	183.36	0.3633	—331.3	—125.6	—105.0	—436.3
V_{23}	234.26	66.63	164.46	0.4051	211.71 / 18.45	66.57	253.11	0.2609	—331.5	—167.1	— 87.6	—419.1
V_{24}	412.74	66.57	236.21	0.2818	580.79 / 18.45	66.44	521.80	0.1273	—323.6	—212.2	— 60.9	—384.5
V_{25}	805.44	66.44	502.90	0.1321	3.009 pa				—296.0	—253.9	— 28.5	—324.5

The stresses in the side span are given by $D = Z + H \cdot u$. The Z influence lines for the diagonals of the system D are constructed in Fig. 3, Plate III, with the aid of the intercepts D and D′ laid off on the two reaction verticals. The quantities D and D′ represent the stresses in any member producible by a unit force acting vertically at the left or right end of the span; they are obtained from a force polygon, Fig. 4, Similarly, the stresses u producible by a force of 1 . $sec\ a$ acting along the line joining the two points of support are obtained by a force polygon, Fig. 2. The maximum tension in any diagonal D is obtained when the load covers the portion of the side span to the right of the panel containing the member. With the diagonals D_4, D_5, D_6, D_9 and D_{10}, there must be added to the above a load covering the center span together with the other (right) side span. In Table VIII, the quantities of column 6 are obtained from the influence line for Z, and those of column 5 from the ordinates of the H-curve in the loaded section. From these, the stresses given in the remaining columns are obtained, using the values $pa = 84.37$ tonnes and $\frac{pa}{G} = 0.6967$. The temperature stresses are given by $D_t = 216.16\ u$.

If the other diagonal D', in any panel, is brought into action, then its stresses will be $D' = -D\,\dfrac{d'}{d}$, where d and d' are the lengths of the respective diagonals. We can therefore obtain the maximum stresses in the diagonals of the system D' by using the influence lines drawn for the system D; for this purpose we simply measure the negative portions of these influence areas and multiply the resulting stresses by $\dfrac{d'}{d}$.

The stresses D' in Table VIII are thus obtained from $D' = (Z + H\,u)\,\dfrac{d'}{d}$.

TABLE VIII.

Diagonal Stresses in the Side Spans.

Member	D	D′	u	H (t)	Z (t)	Hu (t)	$\frac{d'}{d}$	Moving Load $Z+Hu$ (t)	Temperature $\pm 40°\ C.$ (t)	L. L. + Temp. (t)
D_2	+1.053		− 0.100	$35.28\dfrac{pa}{G}$	$4.1757pa = 352.3$	−2.4		349.9	21.6	371.5
D_3	+0.710		− 0.037	31.50 "	2.1820 " = 184.1	−0.8		183.3	8.0	191.3
D_4	+0.470	−2.826	+ 0.015	$\left.\begin{matrix}3530.3\text{ "}\\26.87\text{ "}\end{matrix}\right\}$	1.0890 " = 91.9	+37.1		129.0	3.3	132.3
D_5	+0.283	−2.388	+0.0405	$\left.\begin{matrix}3530.3\text{ "}\\21.77\text{ "}\end{matrix}\right\}$	0.4748 " = 40.1	+114.8		154.9	10.1	165.0
D_6	+0.145	−2.034	+0.0705	$\left.\begin{matrix}3530.3\text{ "}\\16.57\text{ "}\end{matrix}\right\}$	0.1669 " = 14.1	+173.9		188.0	15.2	203.2
D_7	+2.040	−0.185	− 0.040	15.66 "	1.8078 " = 152.5	−0.4		152.1	8.7	160.8
D_8	+2.432	−0.305	− 0.014	10.67 "	1.2517 " = 105.6	−0.1		105.5	3.0	108.5
D_9	+3.056	−0.485	+ 0.020	$\left.\begin{matrix}3530.3\text{ "}\\5.95\text{ "}\end{matrix}\right\}$	0.7240 " = 61.1	+49.2		110.3	4.3	114.6
D_{10}	+4.030	−0.760	+ 0.071	$\left.\begin{matrix}3530.3\text{ "}\\1.94\text{ "}\end{matrix}\right\}$	0.2511 " = 21.2	+174.4		195.6	15.3	210.9
D'_2				$\left.\begin{matrix}3530.3\text{ "}\\1.63\text{ "}\end{matrix}\right\}$	0.2516 " = 21.2	+245.6	$\frac{26.02}{24.00}=1.0840$	289.3	23.5	312.8
D'_3				$\left.\begin{matrix}3530.3\text{ "}\\5.41\text{ "}\end{matrix}\right\}$	0.8123 " = 68.5	+91.0	$\frac{28.64}{26.02}=1.1006$	175.6	8.8	184.4
D'_4				10.04 "	1.4327 " = 120.8	−0.1	$\frac{31.87}{28.64}=1.1128$	134.2	3.6	137.8
D'_5				15.14 "	2.1052 " = 177.6	−0.5	$\frac{35.73}{31.87}=1.1212$	198.6	11.3	209.9
D'_6				20.34 "	2.6089 " = 220.1	−1.0	$\frac{40.22}{35.73}=1.1255$	246.6	17.1	263.7
D'_7				$\left.\begin{matrix}3530.3\text{ "}\\21.25\text{ "}\end{matrix}\right\}$	0.3091 " = 26.1	+98.8	$\frac{45.33}{30.52}=1.4852$	185.5	12.9	198.4
D'_8				$\left.\begin{matrix}3530.3\text{ "}\\25.24\text{ "}\end{matrix}\right\}$	0.7012 " = 59.2	+34.6	$\frac{40.56}{26.18}=1.5497$	145.3	4.7	150.0
D'_9				30.96 "	1.4804 " = 124.9	−0.4	$\frac{36.37}{22.83}=1.5933$	198.3	6.9	205.2
D'_{10}				34.97 "	2.9937 " = 252.5	−1.7	$\frac{32.78}{20.52}=1.5974$	400.7	23.9	424.6

The stresses in the *posts* can also be determined from the influence lines constructed for the system of diagonals D. If, with this system acting, $\mathbf{V}_m$ and $\mathbf{V}'_m$ are the stresses produced in the mth post by a unit force applied vertically at the left or right end of the span, then we have

$$V_m = -\frac{\mathbf{V}_m}{\mathbf{D}_m} \cdot D_m$$

for any loading which would stress the diagonals D in the adjoining panels, and

$$V_m = -\frac{\mathbf{V}'_m}{\mathbf{D}_{m+1}} \cdot D_{m+1}$$

for any loading which would stress the diagonals D'.

$\mathbf{V}_m$ and $\mathbf{V}'_m$ are obtained from the force polygon Fig. 4; D_m and D_{m+1} are calculated by the expression $Z + H u$ from the appropriate positive or negative portions of the influence areas for the mth and $(m+1)$th diagonals. These cases are marked in Fig. 3, Plate III, by dash and dot lines. For the members V_1, V_6 and V_{10}, separate influence lines are drawn.

We thus find, for each post, two stress-values (compression), the larger of which is retained. The quantities involved in the calculation are given in Table IX.

TABLE IX.

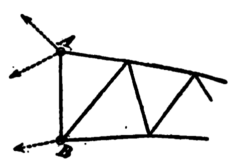
Stresses in the Posts in the Side Spans.

Member	u	H	Z	$Z+Hu$ (t)	$\frac{\mathbf{V}'_m}{\mathbf{D}_m}$ and $\frac{\mathbf{V}'_m}{\mathbf{D}_{m+1}}$	L. L. Stress (t)	Tem. +40° −40° (t)	L. L. + Tem. (t)
V_1	+0.110	$35.24\frac{pa}{G}$	3.2581 pa	−274.9+ 2.7=−272.2	1.00	−272.2	−23.8	−296.0
V_2	+0.100	31.61 "	3.2524 "	−274.4+ 2.2=−272.2	$\frac{0.648}{1.053}$=0.6154	−167.5	−13.3	−180.8
V_3	+0.037	27.10 "	1.6650 "	−140.5+ 0.7=−139.8	$\frac{0.490}{0.710}$=0.6901	− 96.5	− 6.9	−103.4
V_4	− 0.015	3530.3 " } 22.00 "	0.8073 "	− 68.1−37.1=−105.2	$\frac{0.350}{0.470}$=0.7447	− 78.3	− 9.5	
V_5	−0.0465	3530.3 " } 17.59 "	0.3298 "	− 27.8−114.7=−142.5	$\frac{0.226}{0.283}$=0.7986	−113.8	−15.3	
V_6	+0.075	16.09 "	1.8643 "	−157.3+ 0.8=−156.5	1.00	−156.5	−28.9	
V_7	+0.040	12.22 "	1.0631 "	− 89.7+ 0.3=− 89.4	$\frac{1.191}{1.020}$=1.0631	−104.3	− 9.2	−113.5
V_8	+0.014	6.19 "	0.5914 "	− 49.9+ 0.1=− 49.8	$\frac{1.380}{1.216}$=1.1350	− 56.6	− 6.9	
V_9	− 0.020	3530.3 " } 2.04 "	0.1982 "	− 16.7−49.2=− 65.9	$\frac{1.620}{1.528}$=1.0602	− 69.8	−11.4	
V_{10}							−22.0	
V_1							− 6.9	
V_2	− 0.037	3530.3 " } 1.74 "	0.2161 "	− 18.2−90.9=−109.1	$\frac{0.588}{0.710}$=0.8281	− 90.4	− 6.6	
V_3	+0.015	5.61 "	0.6682 "	− 56.4+ 0.1=− 56.3	$\frac{0.418}{0.470}$=0.8900	− 50.1	0	
V_4	+0.0465	10.30 "	1.2285 "	−103.6+ 0.3=−103.3	$\frac{0.268}{0.283}$=0.947	− 97.8	− 2.4	−100.2
V_5	+0.0705	15.27 "	1.7989 "	−151.8+ 0.8=−151.0	$\frac{1.010}{1.016}$=0.994	−150.1	− 8.0	−158.1
V_6	+0.095	20.57 "	2.4530 "	−207.0+ 1.4=−205.6	1.00	−205.6	− 6.9	−212.5
V_7	− 0.014	3530.3 " } 21.43 "	0.5083 "	− 42.9−34.5=− 77.4	$\frac{1.131}{1.216}$=0.930	− 72.0	− 2.8	
V_8	+0.020	26.45 "	1.1238 "	− 94.8+ 0.4=− 94.4	$\frac{1.295}{1.528}$=0.848	− 80.1	0	− 80.1
V_9	+0.071	28.83 "	2.3152 "	−195.3+ 1.4=−193.9	$\frac{1.500}{2.015}$=0.7444	−144.3	− 4.6	−148.9
V_{10}	+0.102	35.08 "	2.9690 "	−250.5+ 2.5=−248.0	1.00	−248.0	− 6.9	−254.9

§ 35. The Framed Arch with Fixed Ends or with Double Anchorage.

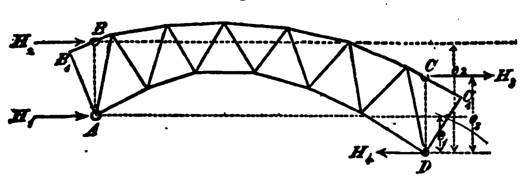
If a framed arch is anchored or supported in two panel-points at each end, then it requires at least 6 conditional equations to fix the reactions. For example, in Fig. 112, if A, B, are the panel-points in which one end of the structure is connected to the abutment, there are required two conditions to fix the reaction at A; while the panel-point B is fixed by the member $A\,B$ and by the plane of the rollers on which it may slide, so that but one additional unknown is introduced at this point. There are thus a total of 6 reaction unknowns for the two ends of the structure; against these we have only the three equations of static equilibrium, so that three more equations of condition must be established from a consideration of the elastic deformations of the system. For the unknowns, to be found from these equations of strain, we may use either the horizontal thrust (i. e., the resultant of the horizontal forces at A and B), the vertical shear and the moment at any cross-section of the arch (e. g., at the crown), or we may consider three of the horizontal reaction-components at the four points of support as our unknowns. The latter scheme will first be applied.

Fig. 112.



1. **General Equations of Condition for the Horizontal Reactions.** If we consider the two vertical members $A\,B$ and $C\,D$ inserted at A and D (Fig. 113), we shall obtain the four

Fig. 113.



panel-points A, B, C, D, which (upon neglecting the slight strains in the members $B\,B_1$ and $C\,C_1$ may be substituted for the actual points of anchorage; at these points, therefore, the horizontal reactions H_1, H_2, H_3 and $H_4 = -(H_1 + H_2 + H_3)$ may be assumed to act. Again let Z denote the stresses which would be produced by the external loading if all the horizontal reactions were equal to zero; in other words, if the structure were held fast at D but free to slide horizontally at A; also, let

u_1 = stresses due to a unit horizontal force at A
u_2 = stresses due to a unit horizontal force at B
z = stresses due to a unit vertical force at A.

Then the stresses in any member of the framed arch would be

$$S = Z + H_1 u_1 + H_2 u_2 - \left(H_1 \frac{e_1}{l} + H_2 \frac{e_2}{l} + H_3 \frac{e_3}{l}\right) z \quad \ldots (474.$$

If $r = \frac{s}{A}$ is the ratio of length to cross-section of a member, then the internal work of deformation for the entire structure is expressed by $\mathbf{W} = \frac{1}{2E} \Sigma\, r\, S^2$; hence the horizontal displacements Δ_1 Δ_2 Δ_3 Δ_4 of the end-points A, B, C, D (considered positive if directed inward) will be determined by the following well-known law (Principle of Virtual Work):

$$\left.\begin{aligned} \Sigma\, r\, S \frac{dS}{dH_1} = (\Delta_1 + \Delta_4)\, E, \qquad \Sigma\, r\, S \frac{dS}{dH_2} = (\Delta_2 + \Delta_4)\, E, \\ \Sigma\, r\, s \frac{dS}{dH_3} = -(\Delta_3 - \Delta_4) E \end{aligned}\right\} \quad \ldots (475,$$

Substituting here the value of S from equ. (474), also putting

$$\frac{dS}{dH_1} = u_1 - \frac{e_1}{l} . z = \mathrm{u}_1$$
$$\frac{dS}{dH_2} = u_2 - \frac{e_2}{l} . z = \mathrm{u}_2$$
$$\frac{dS}{dH} = - \frac{e_3}{l} . z,$$

where u_1 and u_2, denote the stresses produced in the members of the framework by forces of unit horizontal component applied at A and B and acting in the directions $A\,D$ and $B\,D$ respectively, there are obtained the following equations of condition for the three horizontal reactions:

$$\left.\begin{aligned} \Sigma\, r Z \mathrm{u}_1 + H_1 \Sigma\, r \mathrm{u}_1^2 + H_2 \Sigma\, r \mathrm{u}_1 \mathrm{u}_2 - H_3 \frac{e_3}{l} \Sigma\, r z \mathrm{u}_1 = (\Delta_1 + \Delta_4) E \\ \Sigma\, r Z \mathrm{u}_2 + H_1 \Sigma\, r \mathrm{u}_1 \mathrm{u}_2 + H_2 \Sigma\, r \mathrm{u}_2^2 - H_3 \frac{e_3}{l} \Sigma\, r z \mathrm{u}_2 = (\Delta_2 + \Delta_4) E \\ \Sigma\, r Z z + H_1 \Sigma\, r \mathrm{u}_1 z + H_2 \Sigma\, r \mathrm{u}_2 z - H_3 \frac{e_3}{l} \Sigma\, r z^2 = \frac{l}{e_3} (\Delta_3 - \Delta_4) E \end{aligned}\right\} \quad \ldots (476.$$

For abbreviation, let us put

$$\left.\begin{aligned} &\Sigma\, r \mathrm{u}_1^2 = a_1 && \Sigma\, r \mathrm{u}_1 \mathrm{u}_2 = b \\ &\Sigma\, r \mathrm{u}_2^2 = a_2 && \Sigma\, r \mathrm{u}_1 z = c_1 \\ &\Sigma\, r z^2 = a_3 && \Sigma\, r \mathrm{u}_2 z = c_2 \\ &E (\Delta_1 + \Delta_4) - \Sigma\, r\, Z\, u_1 = d_1 \\ &E (\Delta_2 + \Delta_4) - \Sigma\, r\, Z\, u_2 = d_2 \\ &\frac{l}{e_3} E (\Delta_3 - \Delta_4) - \Sigma\, r\, Z\, z = d_3 \end{aligned}\right\} \quad \ldots\ldots\ldots (477.$$

Then the solution of the above equations (476) for the unknowns (H) will give

$$\left.\begin{aligned} H_1 &= \frac{d_1(a_2 a_3 - c_2^2) + d_2(c_1 c_2 - b a_3) + d_3(b c_2 - a_2 c_1)}{a_1 a_2 a_3 + 2 b c_1 c_2 - a_1 c_2^2 - a_2 c_1^2 - a_3 b^2} \\ H_2 &= \frac{d_1(c_1 c_2 - b a_3) + d_2(a_1 a_3 - c_1^2) + d_3(b c_1 - a_1 c_2)}{a_1 a_2 a_3 + 2 b c_1 c_2 - a_1 c_2^2 - a_2 c_1^2 - a_3 b^2} \\ H_3 &= -\frac{l}{e_3}\,\frac{d_1(b c_2 - a_2 c_1) + d_2(b c_1 - a_1 c_2) + d_3(a_1 a_2 - b^2)}{a_1 a_2 a_3 + 2 b c_1 c_2 - a_1 c_2^2 - a_2 c_1^2 - a_3 b^2} \end{aligned}\right\} \quad (478.$$

The series of stresses, u_1 u_2 z, which are needed in making up the quantities a_1, a_2, etc., may be obtained graphically by means of force diagrams. We will again construct influence lines for H, but for this purpose we must first determine the stresses Z produced by different positions of a concentration $P = 1$. With a symmetrical form of arch, it will suffice to obtain merely the stresses z; since, for a load situated at any panel-point distant x and x' from the ends A and D respectively,

$$\Sigma r\, Z\, u = \sum_0^x r\, z\, u + \frac{x}{l} \sum_x^{x'} r\, z\, u$$

$$\Sigma\, r\, Z\, z = \sum_0^x r\, z^2 + \frac{x}{l} \sum_x^{x'} r\, z^2 \ .$$

where $\sum_0^x$ refers to all the members between the reaction A and the load, and $\sum_x^{x'}$ refers to all the members between the load and the symmetrically located panel-point.

Having found the horizontal reactions, the stresses in the members of the framework are determined by equ. (474). The total horizontal thrust transferred to the abutment is

$$H = H_1 + H_2 \ \ldots\ldots\ldots\ldots\ldots\ldots (479.$$

while its position above the point A is

$$e = -\frac{H_2(e_2 - e_1)}{H_1 + H_2} \ \ldots\ldots\ldots\ldots\ldots\ldots (480.$$

and its position above the point D is

$$e' = \frac{H_3 \,.\, e_3}{H_1 + H_2} \ \ldots\ldots\ldots\ldots\ldots\ldots (481.$$

The vertical reaction at the left end of the span will be

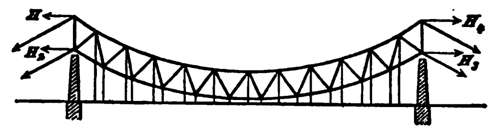
$$V = \mathbf{V} - H_1 \frac{e_1}{l} - H_2 \frac{e_2}{l} - H_3 \frac{e_3}{l} \ldots\ldots\ldots (482.$$

when **V** denotes the reaction for a freely supported beam.

Equations (478) determine the horizontal reactions for the general case, assuming that certain displacements Δ are produced at the abutments. If the abutments are unyielding, then we

must put the displacements $\Delta = 0$ in the expressions for d. The equations are, however, also applicable to the case of yielding points of support such as are involved in the braced suspension bridge outlined in Fig. 114. The elastic displace-

Fig. 114.



ments of the points of support are here governed by the elongations of the anchor chains and these are determined by the following equations in which s_0 is the common length, A_0 the common cross-section and $r_0 = \frac{s_0}{A_0}$:

$$\Delta_1 = -\frac{r_0}{E} H_1, \qquad \Delta_2 = -\frac{r_0}{E} H_2, \qquad \Delta_3 = +\frac{r_0}{E} H_3,$$

$$\Delta_4 = -\frac{r_0}{E}(H_1 + H_2 + H_3)$$

Equations (478) may be applied directly to this case if the coefficients a_1, a_2, ..., are replaced by the following values:

$$\left.\begin{array}{lll} a_1' = a_1 + 2r_0 & b' = b + r_0 & d_1 = -\Sigma r Z u_1 \\ a_2' = a_2 + 2r_0 & c_1' = c_1 - \frac{l}{e_3} r_0 & d_2 = -\Sigma r Z u_2 \\ a_3' = a_3 + 2\frac{l^2}{e_3^2} r_0 & c_2' = c_2 - \frac{l}{e_3} r_0 & d_3 = -\Sigma r Z z \end{array}\right\} \quad (483.$$

To find the stresses produced by temperature variation, we must go back to equation (475) which may now be written

$$\Sigma (rS + E\omega ts)\frac{dS}{dH_1} = E(\Delta_1 + \Delta_4)$$

or

$$\Sigma r S u_1 = E(\Delta_1 + \Delta_4 - \omega t \Sigma s u_1)$$

From this it follows that the stresses accompanying a uniform temperature change may be obtained by equ. (478) if, retaining the other coefficients, we merely put

$$\left.\begin{array}{l} d_1 = -E\omega t \Sigma s u_1 = E\omega t l' \\ d_2 = -E\omega t \Sigma s u_2 = E\omega t l'' \\ d_3 = -E\omega t \frac{l}{e_3} \Sigma s z = 0 \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots \quad (484.$$

Here l' and l'' denote the lengths of the connecting lines AD and BD; and the relations $\Sigma s u_1 = -l'$, $\Sigma s u_2 = -l''$, $\Sigma s z = 0$, are obtained from *Mohr's Theorem*, cited on page 222.

In Plate I are constructed the force polygons of $u_1 = u_1$, u_2 and z for the arch represented in Fig. 5, and the resulting influence lines for H_1 and H_2 are drawn in Fig. 5a. Assuming a uniform cross-section $A_c = 250$ sq. cm for all the chord members and $A_d = 25$ sq. cm. for all the diagonals, there are obtained the expressions

$$H_1 = \frac{149{,}951\, d_1 - 163{,}026\, d_2 + 641\, d_3}{3{,}407{,}067}$$

$$H_2 = \frac{-163{,}062\, d_1 + 239{,}439\, d_2 + 6{,}873\, d_3}{3{,}407{,}067}$$

From these are obtained the following values for the horizontal reactions:

at $\frac{x}{l} =$	$\frac{1}{26}$	$\frac{3}{26}$	$\frac{5}{26}$	$\frac{7}{26}$	$\frac{9}{26}$	$\frac{11}{26}$	$\frac{13}{26}$
$H_1 =$	0.157	0.372	0.565	0.666	0.658	0.612	0.481
$H_2 =$	0.100	0.187	0.272	0.407	0.594	0.734	0.922
$-H_3 =$	0.364	0.782	1.030	1.183	1.187	1.056	0.922
$H_4 =$	-0.107	-0.223	-0.193	-0.110	0.065	0.290	0.481

Using the values $l = 80$ m., $E\,\omega\,t = \pm\ 9600$, and $r = \frac{0.025}{6.2} = 0.004$ for a chord member near the crown, the temperature effects are represented by

$$H_{1,t} = -\frac{13111 \times 0.004}{3{,}407{,}067} E\,\omega\,t\,l = \mp 11.8 \text{ tonnes}$$

$$H_{2,t} = \frac{76377 \times 0.004}{3{,}407{,}067} E\,\omega\,t\,l = \pm 68.7 \text{ tonnes.}$$

2. **A Simplified Method of Design for Braced Arches with Fixed Ends.** This consists in establishing the required equations of condition and thence devising a graphic treatment, exactly as in the case of the plate arch rib with fixed ends. Adopting an arbitrary pair of coordinate axes, with the Y-axis vertical, and defining all the panel-points of the structure by their distances x and y from these axes (where x should be measured horizontally even though the axis of abscissae be inclined), then the bending moment about any panel-point (x, y) may be expressed, as in equation (309), by

$$M = \mathbf{M} - H \cdot y - X_1\, x - X_2,$$

where $\mathbf{M}$ is again the moment for a simple beam, H the resulting horizontal thrust, and, by equ. (308),

$$X_1 = H \cdot \frac{e}{l} \qquad X_2 = H \cdot z_0$$

With less error than in the case of the two-hinged arches, we may here neglect the strains in the web members. The stress in the chord member lying opposite the panel-point (x, y) will be $S = \mp \frac{\mathbf{M}}{h} \sec \sigma$, where the upper sign applies to the upper chord and the lower sign to the lower chord. Considering the system as unstrained and free from temperature stress, the

Principle of the Virtual Work of Deformation yields the following equations of condition

$$\Sigma \frac{S}{EA} \frac{dS}{dH} . s = 0, \qquad \Sigma \frac{S}{EA} \cdot \frac{dS}{dX_1} . s = 0, \qquad \Sigma \frac{S}{EA} \cdot \frac{dS}{dX_2} . s = 0.$$

Observing that

$$\frac{dS}{dH} = \pm \frac{y}{h} sec\,\sigma, \qquad \frac{dS}{dX_1} = \pm \frac{x}{h} sec\,\sigma, \qquad \frac{dS}{dX_2} = \pm \frac{1}{h} sec\,\sigma$$

and using the abbreviation

$$\rho = \frac{s A_0}{h^2 A} sec^2\,\sigma = \frac{a A_0}{h^2 A} sec^3\,\sigma \ldots\ldots\ldots\ldots (485.$$

where A_0 is an arbitrary mean cross-section, then, with a constant E, the above equations of condition become:

$$\begin{aligned} &\Sigma\, \mathbf{M} \rho y - H \Sigma \rho y^2 - X_1 \Sigma \rho x y - X_2 \Sigma \rho y = 0 \\ &\Sigma\, \mathbf{M} \rho x - H \Sigma \rho y x - X_1 \Sigma \rho x^2 - X_2 \Sigma \rho x = 0 \\ &\Sigma\, \mathbf{M} \rho - H \Sigma \rho y - X_1 \Sigma \rho x - X_2 \Sigma \rho = 0 \end{aligned}$$

As the location of the coordinate axes is not governed by any condition, we may so choose these as to simplify the above equations as much as possible. This may be accomplished by making

$$\Sigma \rho x y = 0, \qquad \Sigma \rho x = 0, \qquad \Sigma \rho y = 0, \ldots\ldots (486.$$

which three conditions together with that of a vertical Y-axis, suffice to fix completely the coordinate axes. There is then obtained

$$H = \frac{\Sigma \mathbf{M} \rho y}{\Sigma \rho y^2}, \qquad X_1 = \frac{\Sigma \mathbf{M} \rho x}{\Sigma \rho x^2}, \qquad X_2 = \frac{\Sigma \mathbf{M} \rho}{\Sigma \rho}.$$

If the loading consists of a unit concentration, the numerators of the above expressions assume the familiar form whereby they may be represented as the static bending moment produced at the point of application of the load by the following quantities considered as forces acting at the individual panel-points:

$$\left. \begin{aligned} v &= \rho \,.\, y = \frac{A_0}{A} \cdot \frac{a . y}{h^2} \,.\, sec^3\,\sigma \\ v' &= \rho \,.\, x = \frac{A_0}{A} \cdot \frac{a . x}{h^2} \,.\, sec^3\,\sigma \\ v'' &= \rho = \frac{A_0}{A} \cdot \frac{a}{h^2} \,.\, sec^3\,\sigma \end{aligned} \right\} \ldots\ldots\ldots\ldots (487.$$

We thus obtain the following formulae, identical with equations (318) to (320) of the theory of the Plate-Arch:

$$\left. \begin{aligned} H &= \frac{\mathbf{M}_v}{\Sigma v y} \\ X_1 &= \frac{\mathbf{M}_{v'}}{\Sigma v' x} \\ X_2 &= \frac{\mathbf{M}_{v''}}{\Sigma v''} \end{aligned} \right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (488.$$

There now remains no difficulty in constructing the influence lines for the quantities H, X_1 and X_2, these lines being obtained as the funicular polygons of the panel-loads v, v' and v''. [See Fig. 65, which applies also to the framed arch if the loads v, v', v'' are calculated by the above equations (487) and are considered as acting at the different panel-points.]

The approximation of considering the elongations in the chord members only is generally admissible. Nevertheless we may include the effect of elongations of the web members, as was demonstrated in the case of the two-hinged arch, by applying certain corrections to the panel-loads v. It is evident, however, that the design will thus be rendered more complicated; besides, for this degree of precision, it is preferable to use the more direct method described in part 1 of this section.

In order to calculate the panel-loads v, v', v'', we must first locate the coordinate axes so as to satisfy the conditions expressed by equations (486). Let us first choose any coordinate system, having its origin at A, and let x' y' denote the corresponding coordinates of any panel-point, a and b the coordinates of the origin of the required system of axes, and α the angle between the two X-axes (both Y-axes being vertical); we will then have

$$x = x' - a$$
$$y = y' - b + (a - x') \tan \alpha$$

and, substituting these expressions in equ. (486), we obtain the following values for a. b and $\tan \alpha$:

$$\left.\begin{aligned} a = \frac{\Sigma \rho x'}{\Sigma \rho} \qquad b = \frac{\Sigma \rho y'}{\Sigma \rho} \\ \tan \alpha = \frac{a \Sigma \rho y' - \Sigma \rho x' y'}{a \Sigma \rho x' - \Sigma \rho x'^2} \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots \quad (489.$$

These expressions will evidently become much simplified when the arch has a vertical axis of symmetry. In such case, $\tan \alpha = 0$ (i. e., the X-axis is parallel to the arch-chord),

$$a = \frac{l}{2} \text{ and } b = \frac{\Sigma \rho y'}{\Sigma \rho} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (490.$$

To find the effect of temperature variation, we must again start with the fundamental equations

$$\Sigma \left(\frac{S}{EA} + \omega t\right) s \frac{dS}{dH} = 0, \qquad \Sigma \left(\frac{S}{EA} + \omega t\right) s \frac{dS}{dX_1} = 0,$$
$$\Sigma \left(\frac{S}{EA} + \omega t\right) s \frac{dS}{dX_2} = 0.$$

Assuming the coordinate axes as defined by equ. (486), we obtain

$$H_t = - \frac{E A_0 \omega t \Sigma s \frac{dS}{dH}}{\Sigma \rho y_2}, \quad X_{1t} = - \frac{E A_0 \omega t \Sigma s \frac{dS}{dX_1}}{\Sigma \rho x^2},$$
$$X_{2t} = - \frac{E A_0 \omega t \Sigma s \frac{dS}{dX_2}}{\Sigma \rho}$$

By applying Mohr's Theorem (see page 222), the summations appearing in the numerators of the above expressions may be determined by a simple geometric construction.* If the arch is symmetrical about a vertical center-line, then $\Sigma s \frac{dS}{dX_1} = 0$ and, very nearly, $\Sigma s \frac{dS}{dH} = -l$, $\Sigma s \frac{dS}{dX_2} = 0$, so that the horizontal thrust due to temperature effect will be

$$H_t = \frac{E A_0 \omega t l}{\Sigma \rho y^2} \qquad (491.$$

and its line of action will coincide with the X-axis defined by equ. (490).

3. **Calculation of the Deflections.** The vertical deflection of any panel-point D caused by any loading which produces the stresses S in the individual members of the framework, will be

$$\Delta y = \Sigma \frac{r}{E} S \,.\, z_x \qquad (492.$$

where z_x denotes the stresses producible in the structure, were it freely supported, by a unit load applied at D. If the external loading consists merely of a concentration P at the point ξ, then

$$S = P\left[z_\xi + H_{1\xi}\, u_1 + H_{2\xi}\, u_2 - H_{3\xi} \frac{e_3}{l} z\right]$$

Similarly, the horizontal displacement of D is given by

$$\Delta x = \Sigma \frac{r}{E} \,.\, S \,.\, w_x \qquad (493.$$

where w_x represents the stresses producible in the structure, were it freely supported, by a unit load applied horizontally at D.

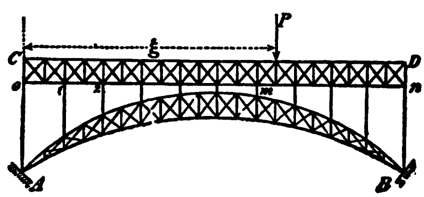
* See the article (previously cited) by *Müller-Breslau* in the Zeitschr. d. Arch. u. Ing. Ver. zu Hannover, 1884.

E. COMBINED SYSTEMS.

§ 36. Combination of Arched Rib with Straight Truss.

1. **Exact Method.** Let a rigidly constructed arch rib, $A\ B$ (Fig. 115), be connected with a straight truss, $C\ D$, by means of the verticals 0, 1, 2, ... n. Let the loading consist of a concentration P bearing directly upon the truss; this load will be partially transmitted to the arch by the verticals. The compressions in these verticals, designated by V_0, V_1, V_2... V_n,

Fig. 115.



will be the only loads acting upon the arch; while the truss will be loaded by the concentration P and the forces $-V_0$, $-V_1$, $-V_2$, etc. There is no difficulty, now, in determining the vertical deflection of the arch-rib at the mth vertical.

In general, let δ_{mr} represent the deflection at the mth panel-point produced by a unit load at panel-point r. By Maxwell's Theorem, this is known to be equal to the deflection producible at the rth panel-point by a unit load acting at panel-point m.

The influence number (δ) may be calculated for a plate-arch by the following expression based on equations (295) and (304):

$$\delta_{mr} = \frac{1}{E\,I'}\,[\mathrm{m} - H_r\,m_m + H_r\,c\,x\,(l - x)]\,\ldots\,(494.$$

and, for a framed arch, by the following expression derived from equ. (460^a):

$$\delta_{mr} = \frac{1}{E}\,[\Sigma\,r\,z_m\,z_r - H_m\,H_r\,\Sigma\,r\,u^2]\,\ldots\ldots\ldots(494^a.$$

With given or assumed sectional areas of the arch, these deflections may at once be determined, since the reactions H produced by the unit load may be computed exactly as for

an independent arch. The total deflection at the mth panel-point caused by the loads V will then be

$$\Delta y_m = V_1 \delta_{1m} + V_2 \delta_{2m} + \ldots + V_m \delta_{mm} + \ldots + V_{n-1} \delta_{(n-1)m}.$$

In like manner, we may compute the deflection at the mth panel-point for the straight truss, considering its ends as simply supported. If ϵ_{mr} denotes this deflection for a unit load at the rth panel-point, its value may be calculated for a plate arch, or approximately for a framed arch, by the formula

$$\epsilon_{mr} = \frac{1}{E I''} m, \qquad \text{(495.}$$

and more accurately for a framed arch by

$$\epsilon_{mr} = \frac{1}{E} \Sigma \, r \, z'_m z'_r, \qquad \text{(495}^a\text{.}$$

where z'_m and z'_r again represent the stresses in the truss produced by unit loads placed at m and r respectively. The total deflection of the truss at panel-point m, produced by the loads $-V$ and $+P$, will then be

$$\Delta y'_m = -(V_1 \epsilon_{1m} + V_2 \epsilon_{2m} + \ldots + V_m \epsilon_{mm} + \ldots + V_{n-1} \epsilon_{(n-1)m}) + P \epsilon_{\xi m}.$$

Also, let $\phi_m V_m$ represent the shortening of the vertical m due to its compressive stress V_m, where ϕ_m may be calculated in the familiar manner when the length and cross-section of the member are given. We then have $\Delta y + \phi_m V_m = \Delta y'$; applying this relation to each separate vertical, and introducing the abbreviated notation

$$\delta_{mr} + \epsilon_{mr} = a_{mr} = a_{rm} \qquad \text{(496.}$$

we obtain the following system of equations:

$$\left.\begin{array}{l}
V_1 (a_{1,1} + \phi_1) + V_2 a_{1.2} + V_3 a_{1.3} + \ldots + V_m a_{1,m} + \ldots \\
\qquad + V_{n-1} a_{1(n-1)} = P \cdot \epsilon_{1,\xi} \\
V_1 a_{1.2} + V_2 (a_{2.2} + \phi_2) + V_3 a_{2,3} + \ldots + V_m a_{2,m} + \ldots \\
\qquad + V_{n-1} a_{2(n-1)} = P \cdot \epsilon_{2.\xi} \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
V_1 \cdot a_{1,m} + V_2 \cdot a_{2,m} + V_3 \cdot a_{3,m} + \ldots + V_m (a_{m,m} + \phi_m) + \ldots \\
\qquad + V_{n-1} a_{m(n-1)} = P \cdot \epsilon_{m\xi} \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
V_1 \cdot a_{(n-1)1} + V_2 \cdot a_{(n-1)2} + V_3 a_{(n-1)3} + \ldots + V_m \cdot a_{(n-1)m} + \ldots \\
\qquad + V_{n-1} [a_{(n-1)(n-1)} + \phi_{n-1}] = P \cdot \epsilon_{(n-1)\xi}
\end{array}\right\} \qquad \text{(497.}$$

These $(n-1)$ equations suffice to determine the $(n-1)$

forces V. We can then obtain the horizontal thrust of the arch by means of the expression

$$H = H_1 V_1 + H_2 V_2 + \ldots + H_m V_m + \ldots + H_{n-1} V_{n-1} \ldots (498$$

Using the above equations, we will construct the influence lines for V_1, V_2, etc., and for H; these will give, for any loading, the values of the external forces for the arch rib as well as for the truss. If M_v denotes the bending moment at any section x of a simple straight beam loaded with the forces V, then the bending moment in the arch will be

$$M'_x = M_v - H \cdot y \ldots \ldots \ldots \ldots \ldots \ldots (499.$$

and the bending moment in the straight truss will be

$$M''_x = P \cdot \frac{(l - \xi)\, x}{l} - M_v \ldots \ldots \ldots \ldots (500.$$

2. **Simplified, Approximate Method of Design.** By the results of § 19, if we neglect the effect of the axial compression, the deflection of the arch at any section distant x_1 from the abutment may be calculated by the following expression (applicable also to framed arches),

$$\Delta y = \frac{1}{E}\left[\frac{x_1}{l}\int_0^l \frac{M'_x}{I'}(l - x)\, dx - \int_0^{x_1} \frac{M'_x}{I'}(x_1 - x)\, dx\right]$$

where M'_x is the bending moment in the arch-axis, and I' denotes the moment of inertia of the arch cross-section multiplied by $\cos \tau = \frac{dx}{ds}$. Similarly we obtain the deflection of the truss:

$$\Delta y' = \frac{1}{E}\left[\frac{x_1}{l}\int_0^l \frac{M''_x}{I''}(l - x)\, dx - \int_0^{x_1} \frac{M''_x}{I''}(x_1 - x)\, dx\right],$$

where M'' and I'' are the bending moment and moment of inertia, respectively, for the given section of the truss. Neglecting, further, the shortening of the verticals, we have $\Delta y = \Delta y'$, or

$$\frac{x_1}{l}\int_0^l \left(\frac{M'_x}{I'} - \frac{M''_x}{I''}\right)(l - x)\, dx - \int_0^{x_1}\left(\frac{M'_x}{I'} - \frac{M''_x}{I''}\right)(x_1 - x)\, dx = 0.$$

The left member of the above equation, however, represents the bending moment at the section x_1 of a simply supported beam which is loaded at each point with a force of magnitude $\left(\frac{M'_x}{I'} - \frac{M''_x}{I''}\right)$. This bending moment must therefore vanish for all the sections x_1, i. e., for all the points at which there is a connection between arch and truss. If we imagine these

connecting verticals to become infinitely close together, then the above condition must hold true for every point of the span; this cannot possibly occur unless all the loads upon the structure reduce to zero, in other words we must have

$$\frac{M'_x}{I'} - \frac{M''_x}{I''} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (501.$$

Hence, the bending moments in the arch and the truss will bear the same proportion as the respective moments of inertia. If these moments of inertia are constant, then the two bending-moments will also have a constant ratio. If this ratio is

$$\frac{I'}{I''} = i,$$

if $\mathbf{M}$ is the moment producible in a simple beam by the external loading, and if M_v, as above, is the moment producible by the loads V, then, by equs. (500) and (499),

$$M''_x = \mathbf{M} - M_v,$$

and

$$M'_x = i\,M''_x = M_v - Hy.$$

Hence

$$M''_x = \frac{\mathbf{M} - Hy}{1+i}, \quad M'_x = \frac{i}{1+i}(\mathbf{M} - Hy) \ldots (502.$$

To evaluate H, neglecting the effect of the axial compression, we use the equation

$$\Delta l = \frac{i}{1+i}\int_0^l \frac{M'_x}{EI'}\,y\,dx = \left(\frac{i}{1+i}\right)^2 \int_0^l \frac{\mathbf{M} - Hy}{EI'}\,y\,dx,$$

whence

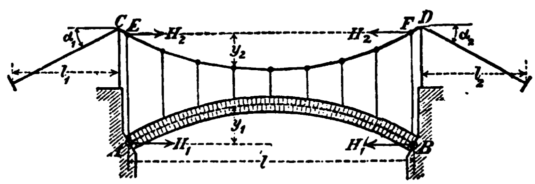
$$H = \frac{\int_0^l \frac{\mathbf{M}}{EI'}\,y\,dx - \frac{(1+i)^2}{i^2}\,.\,\Delta l}{\int^l \frac{y^2}{EI'}\,dx} \ldots\ldots\ldots (503.$$

Omitting the end-displacements from consideration, the horizontal thrust assumes exactly the same value as in the directly-loaded arch without a stiffening truss. Also, the critical loads and maximum moments are to be found in the same manner as for the simple structure; but the actual moments in the arch and the truss will be respectively $\frac{i}{1+i}$ and $\frac{1}{1+i}$ times the corresponding moments for the directly loaded arch.

§ 37. Combination of Arched Rib and Cable.

Let the cable, supported at C and D on rollers or rocker-arms, be connected by vertical rods with the arch $A\,B$ (Fig. 116). Let the horizontal thrust of the arch be denoted by H_1, and

Fig. 116.



the horizontal tension of the cable by H_2. Finally, let y_1 represent the ordinates of the arch-axis referred to the line joining its end points $(A-B)$, and y_2 the ordinates of the cable-polygon referred to the connecting line $E\,F$. In developing the following theory we assume, as with the systems previously considered, that the deformations are so small as to have but a negligible effect upon the bending moments. Adopting the notation

$\lambda =$ length of cable between panel-points,
$a =$ distance between two suspension rods,
$A_c =$ cable cross-section at the crown,
$A_a =$ average cross-section of the arch,
$A_s =$ cross-section of a suspension rod,
$I_1 = I\cos\phi =$ moment of inertia of an arch-section multiplied by the cosine of the inclination of the arch-axis,
$b =$ total length of the arch-axis,

then the stress in the cable section will be

$$T = H_2\frac{\lambda}{a}$$

and the tension in a suspension rod will be

$$S = H_2\frac{\Delta^2 y_2}{a}$$

Noting that the moment of the suspension forces at the section $x\,y_1$ of the arch $= H_2\,y_2$, and if $\mathbf{M}$ represents the moment

of the applied loading in a simple beam of span l, then the bending moment in the arch is found to be

$$M = \mathbf{M} - H_1 \cdot y_1 - H_2 \cdot y_2 \quad \ldots\ldots\ldots\ldots\ldots (504.$$

The axial thrust in the arch may, with sufficient accuracy, be assumed equal to H_1. Assuming, furthermore, that the anchored ends of the cable are displaced through δ_1 and δ_2 in the direction of the backstays, that the supports at C and D are forced together through δ_3 and δ_4 respectively, and that, by a yielding of the abutments, the span of the arch is increased by Δl; then, if the temperature differs from that of the unstressed condition by $t°$, the Principle of Virtual Work (Castigliano's Theorem) yields the following two equations:

$$-\Delta l = \frac{\partial W}{\partial H_1} = \int_0^l \frac{M}{E I_1} \frac{\partial M}{\partial H_1} \cdot dx + \frac{H_1 b}{E A_a} - \omega t b$$

$$-\delta_1 \sec \alpha_1 - \delta_2 \sec \alpha_2 - \delta_3 (\tan \alpha_1 + \tan \alpha'_1) - \delta_4 (\tan \alpha_2 + \tan \alpha'_2)$$
$$= \frac{\partial W}{\partial H_2} = \int \frac{M}{E I_1} \frac{\partial M}{\partial H_2} \cdot dx + \frac{H_2}{E A_c} \Sigma \frac{\lambda^2}{a} + \frac{H_2}{E A_c} (l_1 \sec^2 \alpha_1 + l_2 \sec^2 \alpha_2)$$
$$+ \frac{H_2}{E A_s} \Sigma \left(\frac{\Delta^2 y_2}{a}\right)^2 . s + \omega t \left(\Sigma \frac{\lambda^2}{a} + l_1 \sec^2 \alpha_1 + l_2 \sec^2 \alpha_2 + \Sigma \frac{\Delta^2 y_2}{a} . s\right)$$

or, using the abbreviation

$$\delta_1 \sec \alpha_1 + \delta_2 \sec \alpha_2 + \delta_3 (\tan \alpha_1 + \tan \alpha'_1) + \delta_4 (\tan \alpha_2 + \tan \alpha'_2) = \Delta k,$$

and substituting the value given by equ. (504) for M,

$$\left.\begin{aligned}
&H_1 \left[\int_0^l \frac{y_1^2 dx}{I_1} + \frac{b}{A_a}\right] + H_2 \int_0^l \frac{y_1 y_2}{I_1} dx \\
&\qquad = \int_0^l \frac{\mathbf{M} y_1 dx}{I_1} - E . \Delta l + E \omega t b \\
&H_1 \int_0^l \frac{y_1 y_2}{I_1} . dx + H_2 \left[\int_0^l \frac{y_2^2 dx}{I_1} + \frac{1}{A_c} \left(\Sigma \frac{\lambda^2}{a} + l_1 \sec^2 \alpha_1\right.\right. \\
&\quad \left.\left. + l_2 \sec^2 \alpha_2\right) + \frac{1}{A_s} \Sigma s \left(\frac{\Delta^2 y_2}{a}\right)^2\right] = \int \frac{\mathbf{M} y_2}{I_1} dx - E \Delta k \\
&\qquad - E \omega t \left(\Sigma \frac{\lambda^2}{a} + l_1 \sec^2 \alpha_1 + l_2 \sec^2 \alpha_2 + \Sigma \frac{\Delta^2 y_2}{a} s\right)
\end{aligned}\right\} \ldots (505.$$

These two equations serve to determine the two unknowns, H_1 and H_2.

If the external loading consists merely of a concentration, then, as proved in an earlier part of this book, the definite integrals $\int \frac{\mathbf{M} y_1}{I_1} dx$ and $\int \frac{\mathbf{M} y_2}{I_1} dx$ may be conceived as

the moments in a simple beam loaded with the quantities $\frac{y_1}{I_1}$ and $\frac{y_2}{I_1}$, and may be thus represented by means of a funicular polygon.

If we make $y_2 = 0$ in the first of the equations (505), it reduces to the corresponding formula for a simple arch with hinged-ends. Making $y_1 = 0$ in the second equation, there results the approximate formula for the horizontal tension in a cable connected with a straight stiffening truss.

If both arch and cable have an initial parabolic form, with versines f_1 and f_2 respectively, then, for a constant moment of inertia in the arch,

$$\int_0^l \frac{y_1^2 dx}{I_1} = \frac{8 f_1^2 l}{15 I_1}, \quad \int_0^l \frac{y_2^2 dx}{I_1} = \frac{8 f_2^2 l}{15 I_1}, \quad \int \frac{y_1 y_2}{I_1} dx = \frac{8 f_1 f_2 l}{15 I_1}.$$

Furthermore, for a concentration P acting at a distance ξ from the end A,

$$\int_0^l \frac{\mathbf{M} y_1 dx}{I_1} = \frac{f_1 \xi (l-\xi)(l^2 + l\xi - \xi^2)}{3 l^2 I_1} P,$$

$$\int_0^l \frac{\mathbf{M} y_2 dx}{I_1} = \frac{f_2 \xi (l-\xi)(l^2 + l\xi - \xi^2)}{3 l^2 I_1} P.$$

Putting

$$k = \Sigma \frac{\lambda^2}{a} + l_1 \sec^2 a_1 + l_2 \sec^2 a_2 + \frac{A_c}{A_s} \Sigma s \left(\frac{\Delta^2 y_2}{a}\right)^2,$$

neglecting the term representing the elongations in the rods, and assuming a continuous curvature of the cable, there results

$$k = l\left(1 + \frac{16}{3}\frac{f_2^2}{l^2}\right) + l_1 \sec^2 a_1 + l_2 \sec^2 a_2 \ldots . \quad (506.$$

Using this value, also putting $\Delta l = \Delta k = 0$ and $t = 0$, the solution of equations (505) will give

$$\left.\begin{aligned} H_1 &= \frac{5}{8} \frac{P \cdot \xi \cdot (l-\xi)(l^2 + l\xi - \xi^2)}{f_1 l^3 \left[1 + \frac{b f_2^2}{k f_1^2}\frac{A_c}{A_s} + \frac{15}{8}\frac{b}{l}\frac{I_1}{A_s f_1^2}\right]} \\ H_2 &= \frac{5}{8} \frac{P \cdot \xi (l-\xi)(l^2 + l\xi - \xi^2)}{f_2 l^3 \left[1 + \frac{k f_1^2}{b f_2^2}\frac{A_s}{A_c} + \frac{15}{8}\frac{k}{l}\cdot\frac{I_1}{A_c f_2^2}\right]} \end{aligned}\right\} \ldots (507.$$

The two horizontal forces therefore have a constant ratio, namely,

$$\frac{H_1}{H_2} = \frac{k f_1 A_s}{b f_2 A_c} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (508.$$

This ratio will hold true for any arbitrary loading.

The effect of temperature, together with a yielding of the abutments and anchorages, will produce horizontal forces given by the following formulae:

$$\left.\begin{aligned} H_{1,t} &= \frac{E A_c f_2^2 \left[\left(1+\frac{15}{8}\frac{I_1}{A_c f_2^2}\cdot\frac{k}{l}\right)(\omega t b - \Delta l) + \frac{f_1}{f_2}(\omega t k + \Delta k)\right]}{k f_1^2 \left(1+\frac{b f_2^2}{k f_1^2}\frac{A_c}{A_a}+\frac{15}{8}\frac{b}{l}\frac{I_1}{A_a f_1^2}\right)} \\ H_{2,t} &= \frac{E A_a f_1^2 \left[\left(1+\frac{15}{8}\frac{I_1}{A_a f_1^2}\cdot\frac{b}{l}\right)(\omega t k + \Delta k) + \frac{f_2}{f_1}(\omega t b - \Delta l)\right]}{b f_2^2 (1+\frac{k f_1^2}{b f_2^2}\frac{A_a}{A_c}+\frac{15}{8}\frac{k}{l}\cdot\frac{I_1}{A_c f_2^2}} \end{aligned}\right\} \quad (509.$$

If we put $H_2 = C \,.\, H_1$, so that, by equ. (508),

$$C = \frac{b f_2 A_c}{k f_1 A_a},$$

the bending moment at the section $x y_1$ of the arch may be calculated by the relation

$$M = \mathbf{M} - H_1 (y_1 + C y_2) \quad \ldots\ldots\ldots\ldots\ldots\ldots (510.$$

From this we derive the following rule for finding the maximum moments and critical loads for the arch: Increase the rise (versine) of the arch by the amount $C f_2 = \frac{b f_2 A_c}{k f_1 A_a} \cdot f_2$ and take the resulting parabola as the axis of a new arch for which the maximum moments, etc., are to be determined in the usual manner, using the value of H_1 given by equ. (507).

In calculating the temperature effect, we must keep in mind that a negative value of H_2, with flexible cable and suspenders, cannot be transmitted to the arch. If, upon an increase of temperature, H_2 becomes negative, then the cable becomes ineffective and the entire structure acts merely as a simple two-hinged arch. It can be easily demonstrated that this condition may occur with a quite moderate rise of temperature and that, even with the structure completely loaded, the cable may become slack unless there is a simultaneous yielding of the arch abutments or unless the cable has been very skilfully mounted. The horizontal tension in the cable due to temperature will exceed that due to the loading (of intensity q), when

$$E \omega t \left[\frac{A_a (f_1 k + f_2 b) f_1}{b f_2} + \frac{15}{8}\frac{I_1 k}{f_2 l}\right] > \frac{1}{8} q l^2$$

or, on substituting $I_1 = A_a \frac{h^2}{4}$ and $\sigma' A_a = \frac{1}{8}\frac{q l^2}{f_1}$, when

$$\frac{E \omega t}{\sigma'}\left[1 + \frac{k f_1}{b f_2} + \frac{15}{32}\frac{h_2 k}{f_1 f_2 l}\right] > 1$$

Here σ' represents, approximately, the mean stress at the crown of the arch produced by a full-span load. This stress, in a

wrought iron arch, need not be taken higher than 500 kg. per sq. cm. (= 7120 pounds per square inch), bearing in mind that this does not include the stresses due to temperature, etc. If we put the expression within the brackets approximately =2, the temperature rise need not exceed

$$t = \frac{\sigma'}{2 E \omega} = \frac{500}{2 \times 24} = 10^\circ\ C. \ (= 18^\circ\ F.)$$

before the cable will become slack. In reality, however, we may expect much greater fluctuations of temperature than this, whence we may conclude that the cable will not be effective at all times and that, consequently, *this combination cannot be considered an advantageous form of construction.*

It should also be observed that the versines, f_2 and f_1, of the cable and of the arch, must have a certain ratio in order that the maximum intensities of stress in these two parts of the structure may be approximately equal to each other or to assigned values. Introducing the abbreviation $N = 1 + \frac{b f_2^2 A_c}{k f_1^2 A_a} + \frac{15}{8} \frac{b}{l} \frac{I_1}{A_a f_1^2}$, and neglecting temperature stresses, the maximum stress in the cable becomes

$$\sigma_c = \frac{1}{8} \frac{q l^2}{f_1 N} \cdot \frac{b f_2}{k . f_1 A_a}.$$

Similarly, the maximum stress in the crown-section of the arch will be, when the cable is slack,

$$\sigma_a = \frac{2 q l^2 + 15 E \omega t A_a . h}{16 A_a f_1 \left(N - \frac{b f_2^2 A_c}{k f_1^2 A_a}\right)}.$$

We therefore have

$$\frac{\sigma_a}{\sigma_c} = \left(1 + \frac{15}{16} \frac{E \omega t}{\sigma'} \frac{h}{f_1}\right) \frac{N}{N - \frac{b f_2^2 A_c}{k f_1^2 A_a}} \cdot \frac{k f_1}{b f_2}$$

or approximately

$$\frac{\sigma_a}{\sigma_c} = \left(1 + \frac{15}{16} \cdot \frac{960}{500} \cdot \frac{h}{f_1}\right) 2 \cdot \frac{k f_1}{b f_2}.$$

If arch and cable are to be equally stressed, we must accordingly have

$$\frac{f_1}{f_2} = \frac{b}{2 k} \cdot \frac{1}{1 + 1.8 \frac{h}{f_1}}$$

or the rise of the arch must be less than $\frac{1}{2}$ to $\frac{1}{3}$ the versine of the cable.

APPENDIX.

The Elastic Theory Applied to Masonry and Concrete Arches.

The theory of arches developed above, in §§ 10 to 24, based on the elastic deformations, makes no other assumption in regard to the material of construction than that it is to be considered perfectly elastic within the limits of stress occurring in the structure. If this assumption is equally applicable to masonry as to metallic structures, the above theory of the solid, elastic arch rib may be as appropriately applied to *masonry arches;* and it need only be observed that, on account of the small tensile strength of the material of a masonry arch, no large tensile stresses may occur, so that wherever such stresses are indicated by the computations the entire cross-section of the arch is not to be considered effective.

The elastic behavior of masonry, previously maintained by *Tredgold* and *Bevan,* appears to be established beyond question by the tests of *Bauschinger**, *Foeppl* and others. All natural and artificial building stones, within the practical limits of stress, are to be regarded as more or less perfectly elastic bodies; and the conclusion is therefore justified that the distribution of normal pressures in any smooth, plane joint in masonry is determined by the elastic law or, with the simplifying assumptions of Navier's hypothesis, by the rules expressed by equations (174) in § 11. An arch composed entirely of squared stone, i. e., without any mortar, will therefore act as an elastic arch, at least so long as no tensile stresses occur in the planes of the joints and the deformations do not exceed certain limits. Masonry composed of stone and mortar, however, on account of its heterogeneity, cannot be treated as perfectly elastic and will be affected not only by the manner of construction but also by the age of the masonry. The more recent the work, the larger will be the ratio of the permanent set to the total deformation. Furthermore, there does not appear to be a definite elastic limit for such masonry. Nevertheless laboratory investigations as well as observations on actual structures† indicate that within certain limits of stress even

*Mitteilungen aus dem mechanisch-technischen Laboratorium zu Munchen, No. X.

†*Köpcke,* "Die Messung von Bewegungen an Bauwerken mit der Libelle." Protokolle des sächs. Ing. Ver., 1877.—"Arc d'experience," Report of *M. de Perrodil.* Ann. des ponts et chaussées, 1882.—"Gewölbe-Versuche des österreichischen Ingenieur- und Architekten-Vereins." Zeitschrift of this Society, 1895; also published as a separate reprint.

mortar masonry acts as an elastic body. Furthermore, even with conditions of imperfect elasticity where permanent compression accompanies the temporary strains, the distribution of pressure in plane joints is really given by the same laws as in perfectly elastic materials, provided we may only assume that the conditions are uniform throughout the mass of masonry and that the permanent strains are proportional to the normal pressures. In fact, as *Foeppl*† has shown, we arrive at results practically identical with those of an elastic arch if we treat the voussoirs as incompressible bodies and ascribe elastic compressibility only to the mortar. Actually, however, the relations are not so simple. The masonry at different points may have varying elasticity and, with uneven hardness, varying compressibility; furthermore, in masonry of unsquared stones, the joints are not true planes. The perfect elastic behavior may therefore best be expected in the case of monolithic concrete arches; nevertheless even for this material, according to the experiments of *Hartig, Bauschinger, von Bach,* and others, *Hooke's* law of proportionality is but approximately true and only within low limits of stress, since for larger stresses the compression increases more rapidly than the pressure. On account of these considerations, the elastic theory is not as accurate a treatment for determining the stresses in masonry arches as in metallic arches; nevertheless, on the basis of the above-mentioned investigations, especially the tests on arches by the Austrian Society of Engineers and Architects, the results of the elastic theory may also be applied to masonry arches thereby arriving at the actual relations in accord with all preceding theories for the masonry arch. The perception that a correct theory for masonry arches should be founded on the laws of elasticity had been expressed by *Poncelet* (1852), but later writers (*Winkler, Culmann, Schwedler* and others) were the first to develop this idea and to apply it to the actual construction of such a theory. Now the theory of the elastic arch is generally recognized as the basis for a correct analysis of the masonry arch, and the methods of design developed in the preceding chapters may therefore be directly applied to masonry arches.

The static analysis of a masonry arch involves the finding of the lines of resistance for specified cases of loading and, ultimately, the determination of the maximum stresses in the joints, which may be found either directly or by means of the line of resistance belonging to the corresponding severest condition of loading. For the rigid solution of this problem, therefore, two cases of loading must be considered for each joint, one for the upper and lower core-point respectively; nevertheless these may be replaced for each joint, without great error,

†*Foeppl,* "Theorie der Gewölbe." Leipzig, 1881.

by a single case of loading corresponding to the middle point of the joint. (Compare the remark in § 29, p. 208.)

If R is the resultant force acting at any section (or joint) of the arch, N its component normal to the section and M its moment referred to the center of gravity of the section (arch-axis), then, with A as the area and Z as the section modulus of the cross-section, the extreme fiber stresses will be

$$\sigma = \frac{N}{A} \pm \frac{M}{Z}.$$

If N and M are taken for a unit width of the arch, then, with d as the thickness of the arch,

$$\sigma = \frac{N}{d} \pm \frac{6M}{d^2}$$

It is advisable to determine the stresses due to the dead and live loads separately, combine these and then, in the case of hingeless arches, add the stresses due to temperature variation.

If the arch is provided with *hinges* at the ends and at the crown, so that it reduces to the statically determinate case of the three-hinged arch, there is no special difficulty in determining the forces acting; for each section to be investigated the critical loading may readily be determined and used as a basis for evaluating the maximum live load stresses in the extreme fibers. For this purpose either the analytical or graphic procedures given in § 15 may be used. If p is the live load per linear foot and if the length of loading is determined for the middle point of each section, then, for a parabolic arch, also approximately for a flat segmental arch, we have the expressions

$$M = \frac{px(l-x)\ (l-2x)}{2(3l-2x)}$$

$$N = \frac{l^3 + 8f^2l\ (l-2x)\ (5l-4x)}{4fl\ (3l-2x)^2} \cdot p \cdot \cos\phi.$$

Here ϕ is the angle of inclination of the section from the vertical and x the distance of its middle point from the end of the span.

If we are dealing with concentrated loads, it is best to employ the method of influence lines. The effect of the dead load is obtained by drawing the corresponding line of resistance or, for greater accuracy, analytically from the relations

$$M = \mathbf{M} - H.\,y \quad \text{and} \quad \mathrm{N} = H.\cos\phi + V.\sin\phi$$

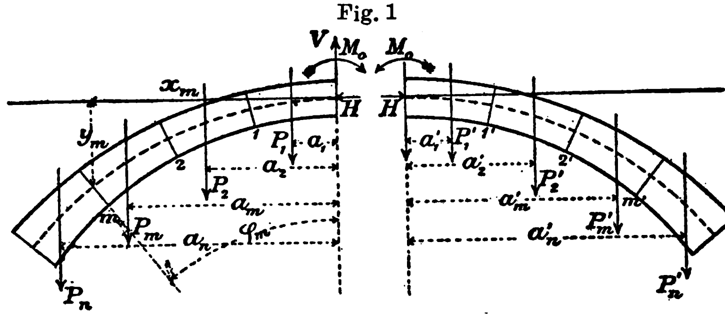
where H is found from the crown moment $\mathbf{M}_c$ of the dead load as $H = \frac{\mathbf{M}_c}{f}$.

Far more detailed and tedious is the correct design of the masonry arch constructed *without hinges,* which is, of course,

the common arrangement. In such an arch, if the applied loading and the accompanying secondary effects produce no tensile stresses in the end and intermediate joints exceeding the tensile strength of the masonry, the arch will act exactly as a hingeless elastic rib and the theory given in §§ 20 and 21 may be directly applied. Here also, to be exact, it would be necessary to find the most unfavorable loading for each section to be investigated and then determine the corresponding stresses. On account of the tediousness of such a design, however, it is frequently deemed sufficient to consider just one or two special cases of loading, usually a loading over the entire arch and one covering the half span. The former gives the maximum stresses in the joints near the crown of the arch, the latter corresponds approximately to the maximum stresses at the quarter points and ends of the span.

Accordingly the external forces together with the pressures in the joints, for each of the three conditions of loading, namely, a.) dead load, b.) full live load and c.) half-span live load,

Fig. 1



may be determined directly without the necessity of first finding the influence lines.

The dead load is composed of the weight of the arch and the pressure of the construction resting upon it. The latter, in the case of arches with spandrel filling, may be taken as equal at each point to the weight of the superincumbent filling; but in arches with large rise and large depth of filling this pressure is to be determined in accordance with the theory of earth pressures. In the case of secondary arches or piers resting upon the main arch, the weight they carry is applied in the form of concentrated loads. Continuous weights and loading, however, will also be replaced by concentrations since we will divide the arch into segments and consider their weights applied at their respective centers of gravity.

a.) *Analytical treatment for a given case of loading.* We first have to determine the resultant force at any one joint; with this known, the pressures at all the remaining joints are readily found. To completely determine such a force, however, three

things must be known: the components normal and parallel to the section and the moment about the axis of the arch. To establish the corresponding equations of condition, the three derivatives of the work of deformation will be employed.

In the following, we assume a masonry arch symmetrical about the crown and adopt as our unknowns the data for the thrust at the crown of the arch, namely, its horizontal and vertical components H and V and its moment M_0 about the center of gravity of the section. In addition, we divide up the arch ring by sections normal to the axis of the arch and uniformly spaced along its length (Fig. 1) and calculate the weight of each segment of the arch together with the superincumbent load. Let these applied forces in the left half of the arch be $P_1, P_2, P_3, \ldots P_n$, and in the right half $P'_1, P'_2, P'_3, \ldots P'_n$; their horizontal distances from the crown joint $a_1, a_2 \ldots$ and $a'_1, a'_2, \ldots$, respectively. The axial points of the cross-sections are referred by coordinates (x, y) to a system of axes passing through the crown of the axis of the arch. The axial thrust and moment for any section inclined at the angle ϕ_n from the normal are then calculated as follows:

$$\left.\begin{array}{l}
\text{In the left half of the arch:} \\
N_m = H \cdot \cos\phi_m - V \cdot \sin\phi_m + (P_1 + P_2 + \ldots + P_m)\sin\phi_m \\
M_m = M_0 - H \cdot y_m - V \cdot x_m + P_1(x_m - a_1) \\
\qquad + P_2 \cdot (x_m - a_2) + \cdots + P_m \cdot (x_m - a_m) \\
\text{In the right half of the arch:} \\
N'_m = H \cdot \cos\phi_m + V \cdot \sin\phi_m + (P'_1 + P'_2 + \ldots + P'_m) \cdot \sin\phi_m \\
M'_m = M_0 - H \cdot y_m + V \cdot x_m + P'_1(x_m - a'_1) + P'_2(x_m - a'_2) \\
\quad + \ldots + P'_m(x_m - a'_m)
\end{array}\right\} \quad (511.$$

With the notation previously employed, the approximate expression for the work of deformation of the arch is

$$W = \frac{1}{2}\int \frac{M^2}{EI} \cdot ds + \frac{1}{2}\int \frac{N^2}{EA} \cdot ds.$$

Hence, with the assumption of perfect fixedness and rigidity of the end sections, we obtain the following equations of condition expressing the Theorem of Least Work:

$$\left.\begin{array}{l}
E\dfrac{dW}{dH} = \displaystyle\int \frac{M}{I} \cdot \frac{dM}{dH} \cdot ds + \int \frac{N}{A} \cdot \frac{dN}{dH} \cdot ds = 0, \\
E\dfrac{dW}{dM_0} = \displaystyle\int \frac{M}{I} \cdot \frac{dM}{dM_0} \cdot ds = 0, \\
E\dfrac{dW}{dV} = \displaystyle\int \frac{M}{I} \cdot \frac{dM}{dV} \cdot ds + \int \frac{N}{A} \cdot \frac{dN}{dV} \cdot ds = 0,
\end{array}\right\}$$

where the integrals represent a summation over the entire length of the span. If the segments of the arch are assumed of uniform length along the axis of the arch, and if a mean cross-section $A_1, A_2, \ldots$, and a mean moment of inertia $I_1, I_2, \ldots$,

are introduced for each segment, then *Simpson's* Rule may be applied for the above summation. The substitution of the values of M and N, as well as of their differential coefficients, from the above expressions (511) yields the following three equations of condition:

$$\left.\begin{array}{l} \mathrm{a}_1 + \mathrm{b}\,H + \mathrm{c}\,M_0 = 0 \\ \mathrm{a}_2 + \mathrm{c}\,H + \mathrm{d}\,M_0 = 0 \\ \mathrm{a}_3 + \mathrm{e}\,V = 0 \end{array}\right\} \qquad (512.$$

Introducing the abbreviations of notation

$(P_1 + P_2 + \ldots + P_m) \cdot \sin\phi_m = \mathbf{P}_m$ for the left, and $\mathbf{P}'_m$ for the right half of the arch,

$P(x_m = a_1) + P_2(x_m - a_2) + \ldots + P_m(x_m - a_m) = \mathbf{M}_m$ for the left and $\mathbf{M}'_m$ for the right half of the arch,

the coefficients in (512) are given by:

$$\left.\begin{array}{l}
\mathrm{a}_1 = -\frac{1}{2}\Big[4\frac{y_1}{I_1}(\mathbf{M}_1 + \mathbf{M}'_1) + 2\frac{y_2}{I_2}(\mathbf{M}_2 + \mathbf{M}'_2) + 4\frac{y_3}{I_3}(\mathbf{M}_3 \\
\qquad + \mathbf{M}'_3) + \ldots + \frac{y_n}{I_n}(\mathbf{M}_n + \mathbf{M}'_n)\Big] + \frac{1}{2}\Big[4\frac{\cos\phi_1}{A_1}(\mathbf{P}_1 + \mathbf{P}'_1) \\
\qquad + 2\frac{\cos\phi_2}{A_2}(\mathbf{P}_2 + \mathbf{P}'_2) + 4\frac{\cos\phi_3}{A_3}(\mathbf{P}_3 + \mathbf{P}'_3) \\
\qquad + \ldots + \frac{\cos\phi_n}{A_n}(\mathbf{P}_n + \mathbf{P}'_n)\Big] \\
\mathrm{a}_2 = \frac{1}{2}\Big[\frac{4}{I_1}(\mathbf{M}_1 + \mathbf{M}'_1) + \frac{2}{I_2}(\mathbf{M}_2 + \mathbf{M}'_2) + \frac{4}{I_3}(\mathbf{M}_3 + \mathbf{M}'_3) \\
\qquad\qquad + \frac{1}{I_n}(\mathbf{M}_n + \mathbf{M}'_n)\Big] \\
\mathrm{a}_3 = -\frac{1}{2}\Big[4\frac{x_1}{I_1}(\mathbf{M}_1 - \mathbf{M}'_1) + 2\frac{x_2}{I_2}(\mathbf{M}_2 - \mathbf{M}'_2) + 4\frac{x_3}{I_3} \\
\qquad\qquad (\mathbf{M}_3 - \mathbf{M}'_3) + \ldots + \frac{x_n}{I_n}(\mathbf{M}_n - \mathbf{M}'_n)\Big] \\
\mathrm{b} = \Big[4\frac{y_1^2}{I_1} + 2\frac{y_2^2}{I_2} + 4\frac{y_3^2}{I_3} + \ldots + \frac{y_n^2}{I_n}\Big] + \Big[\frac{1}{A_0} + \frac{4\cos^2\phi_1}{A_1} \\
\qquad\qquad + \frac{2\cos^2\phi_2}{A_2} + \frac{4\cos^2\phi_3}{A_3} + \ldots + \frac{\cos^2\phi_n}{A_n}\Big] \\
\mathrm{c} = -\Big[\frac{4y_1}{I_1} + \frac{2y_2}{I_2} + \frac{4y_3}{I_3} + \ldots + \frac{y_n}{I_n}\Big] \\
\mathrm{d} = \Big[\frac{1}{I_0} + \frac{4}{I_1} + \frac{2}{I_2} + \frac{4}{I_3} + \ldots + \frac{1}{I_n}\Big] \\
\mathrm{e} = \Big[4\frac{x_1^2}{I_1} + 2\frac{x_2^2}{I_2} + 4\frac{x_3^2}{I_3} + \ldots + \frac{x_n^2}{I_n}\Big] + \Big[4\frac{\sin^2\phi_1}{A_1} + 2\frac{\sin^2\phi_2}{A_2} \\
\qquad\qquad + 4\frac{\sin^2\phi_3}{A_3} + \ldots + \frac{\sin^2\phi_n}{A_n}\Big]
\end{array}\right\} (513$$

With the values of H, M_0 and V calculated from equations (512), equs. (511) are to be used for determining the axial thrust and moment at the other sections of the arch and the resulting normal stresses.

For a loading symmetrical about the crown, we have $\mathbf{M}' = \mathbf{M}$ and $\mathbf{P}' = \mathbf{P}$; hence $V = 0$, so that only H and M_0 remain to be figured. This condition obtains for the dead load when the roadway is horizontal or symmetrically inclined about the crown of the arch. If we investigate only the three cases of loading mentioned above, then H and M_0 need to be figured but once for the dead load and then for the full live load. For the load over the half-span, we need only find the value of V from the third of equs. (512) using $\mathbf{M}' = 0$, $\mathbf{P}' = 0$, since H and M_0 for this case of loading are just one-half of the corresponding values for the full-span load. The stresses calculated for the dead and live loads separately are then to be added algebraically.

Approximate Formulae. We assume an arch with parabolic axis, having a symmetrical form and carrying a symmetrical, continuous loading which amounts to q_0 per unit length at the crown, q_1 per unit length at the ends, and varies between these values as a parabolic function so that at distance x from the crown, $q = q_0 + (q_1 - q_0)\ \frac{4x^2}{l^2}$. We also assume that $I.\cos\phi = I_0$ is a constant, and substitute $\frac{l}{2A_0}$ for the second summation in the expression for the coefficient $\mathfrak{b}$, along with which the terms in the expression for $\mathfrak{a}_1$ referring to the axial thrust are neglected. Again replacing the summations by definite integrals, we readily obtain

$$\mathfrak{a}_1 = -\frac{1}{32}\,\frac{37q_0 + 5q_1}{3\times5\times7}\cdot\frac{fl^3}{I_0}, \qquad \mathfrak{a}_2 = \frac{1}{16}\,\frac{9q_0 + q_1}{30}\cdot\frac{l^3}{I_0},$$

$$\mathfrak{b} = \frac{1}{10}\,\frac{f^2 l}{I_0} + \frac{l}{2A_0}, \qquad \mathfrak{c} = -\frac{1}{6}\,\frac{fl}{I_0}, \qquad \mathfrak{d} = \frac{1}{2}\,\frac{l}{I_0}.$$

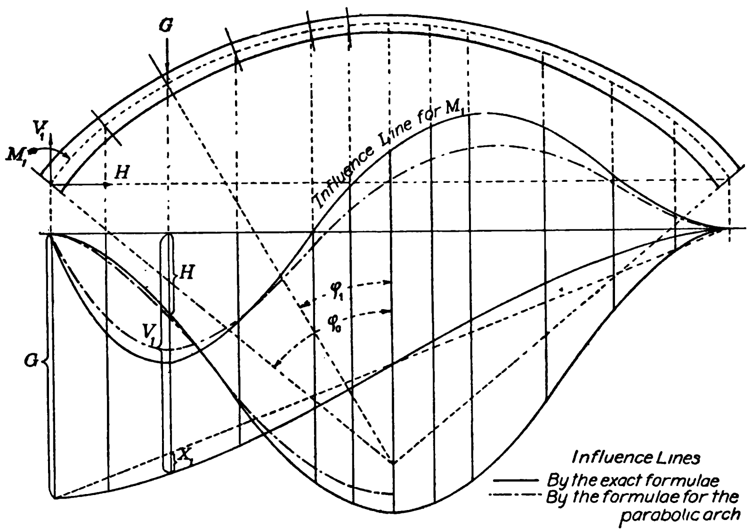
Introducing the symbol $f' = f\left(1 + \frac{45}{4}\,\frac{I_0}{A_0 f^2}\right)$, the solution of equs. (512) yields.

$$\left.\begin{aligned} H &= \frac{6q_0 + q_1}{7}\cdot\frac{l^2}{8f'} \\ M_0 &= \left(\frac{6q_0 + q_1}{7}\cdot\frac{f}{f'} - \frac{9q_0 + q_1}{10}\right)\frac{l^2}{24} \\ &\text{and, for the moment at the ends of the span,} \\ M_n &= -\left[\frac{6q_0 + q_1}{7}\cdot\frac{f}{f'} - \frac{13q_0 - 3q_1}{10}\right]\frac{l^2}{12}. \end{aligned}\right\} \quad \ldots\ldots\ldots (514.$$

These formulae are also applicable to the simplified, approximate design of flat segmental arches. With $q_1 = q_0$, they reduce to the formulae for a load uniformly distributed over the entire span. For a uniform load of p per linear foot over *one-half* of the span, H and Mo take one-half the values given for a full-span load and, in addition, there appears at the crown section a vertical shear (directed upward on the loaded side) of $V = \frac{3}{32}\,pl$.

b.) *Procedure with the aid of influence lines.* If it is desired to investigate various cases of loading, especially the critical loadings for individual sections, it becomes necessary to construct the influence lines for three unknowns of the external forces. For this purpose we may apply the procedure developed in § 20 for the arch with fixed ends, the simplifications there given for parabolic arches with constant moment of inertia being also applicable to flat segmental arches of approximately uniform cross-section.* For arbitrary forms of arches (e. g., basket-handle and equilibrium curves) with variable moments of inertia, there is to be recommended the general graphic method in which the influence lines of the quantities H, X_1 and X_2 are obtained as funicular polygons. For any given loading,

Fig. 2



these quantities are obtained in the familiar manner by summing the ordinates of their influence lines multiplied by the respective loads. With the aid of these quantities, we may then either draw the line of pressures or simply calculate the moments about the core-points of any given section. If x_k and y_k are the coordinates of any core-point referred to the system of axes fixed by equ. (321), then, with the notation of § 20, by equ. (309), $M_k = \mathbf{M}_k - H \, . \, y_k - X_1 . \, x_k - X_2$; from this value of the moment, the stress in the extreme fibers of the cross-section is computed in the ordinary manner. By the same equation, the influence line for M_k may be derived from those for the quantities H,

*On this approximate, simplified design see *Th. Landsberg*, "Beitrag zur Theorie der Gewölbe." Zeitschrift d. Ver. deutscher Ing., 1901.

X_1 and X_2, and, by its aid, the most unfavorable action of the loads may be accurately determined. It will always suffice, however, to use an approximate determination for the lengths of loading giving maximum stress and, for this purpose, we may employ the simplified relations established for the parabolic arch (i. e., determining the limits of loading by equ. (381) or by means of the straight reaction locus and the hyperbolic reaction-envelope curve.)

For the frequently occurring case of a circular arch of large rise-ratio, whose radial thickness increases from the crown to the ends in a uniform ratio with *sec* ϕ, the following may be given in addition to the exact formulae of design.

Adopting the determining quantities of the left end reaction, i. e., H, V_1 and M_1 as the three unknowns, the loading consisting of a single concentration G, and with the notation indicated in Fig. 2, we have the axial thrust and moment at any section given by

$$\left\{\begin{aligned} P_x &= V_1 \cdot \sin\phi + H \cdot \cos\phi - \Big[G \cdot \sin\phi \Big]_{\phi_1}^{\phi} \\ M_x &= M_1 + V \cdot r\,(\sin\phi_0 - \sin\phi) - H \cdot r\,(\cos\phi - \cos\phi_0) \\ &\qquad - \Big[G \cdot r\,(\sin\phi_1 - \sin\phi) \Big]_{\phi_1}^{\phi} \end{aligned}\right.$$

If d_0 is the thickness at the crown, then, for any section, $d = d_0 \sec\phi$ and $I = I_0 . \sec^3\phi = \frac{1}{12} d_0^3 . \sec^3\phi$. Introducing the symbols for abbreviation,

$$\left.\begin{aligned} \delta &= \frac{I_0}{A_0 \cdot r^2} = \frac{1}{12} \cdot \frac{d_0^2}{r^2} \\ \alpha &= (1+\delta)\sin\phi - \frac{1}{3}\sin^3\phi \\ \beta &= \frac{3}{4}\phi + \frac{1}{2}\sin\phi \cdot \cos\phi \cdot \left(\frac{3}{2} + \cos^2\phi\right) \\ \gamma &= \frac{1}{5}\sin\phi\,\left(\cos^4\phi + \frac{4}{3}\cos^2\phi + \frac{8}{3}\right) \end{aligned}\right\} \quad . . (515.$$

and the corresponding symbols $\alpha_0, \beta_0, \gamma_0$, and $\alpha_1, \beta_1, \gamma_1$, for the points $\phi = \phi_0$ and ϕ_1 respectively, we obtain the following expressions from equs. (182) (185) and (186) for the variation of the coordinates of any point in the axis of the arch:

$$\begin{aligned} \Delta\phi - \Delta\phi_0 = -\frac{r}{E\,I_0}\Big[& M_1(\alpha - \alpha_0) + V_1 . r\Big\{ (\alpha-\alpha_0)\sin\phi_0 \\ & + \frac{1}{4}(\cos^4\phi - \cos^4\phi_0)\Big\} + H . r\Big\{ (\alpha - \alpha_0)\cos\phi_0 - \frac{1}{2}(\beta - \beta_0)\Big\} \\ & - G . r\Big\{(\alpha - \alpha_1)\sin\phi_1 + \frac{1}{4}(\cos^4\phi - \cos^4\phi_1)\Big\}\Big] + \omega\, t(\phi - \phi_0) \end{aligned}$$

$$\begin{aligned} \Delta x - \Delta x_0 = -y . \Delta\phi_0 - r \cdot \cos\phi\,(\Delta\phi - \Delta\phi_0) - \frac{r^2}{E\,I_0}\Big[& \frac{1}{2} M_1(\beta - \beta_0) \\ & + V_1 r\Big\{\frac{1}{2}(\beta - \beta_0)\sin\phi_0 + \frac{1}{5}\cos^5\phi - \cos^5\phi_0)\Big\} \end{aligned}$$

$$+H\cdot r\left\{\frac{1}{2}(\beta-\beta_0)\cos\phi_0-(\gamma-\gamma_0)\right\}-G.r\left\{\frac{1}{2}(\beta-\beta_1)\sin\phi_1\right.$$
$$\left.\left.+\frac{1}{5}(\cos^5\phi-\cos^5\phi_1)\right\}\right]$$

$$\Delta y-\Delta y_0=x\cdot\Delta\phi_0-r\cdot\sin\phi\,(\Delta\phi-\Delta\phi_0)-\frac{r^2}{EI_0}\left[\frac{1}{4}M_1\,(\cos^4\phi_0\right.$$
$$-\cos^4\phi)+V_1.r\left\{\frac{1}{4}(\cos^4\phi_0-\cos^4\phi)\sin\phi_0-\frac{1}{5}\sin^3\phi\left(\frac{2}{3}+\cos^2\phi\right)\right.$$
$$\left.+\frac{1}{5}\sin^3\phi_0\left(\frac{2}{3}+\cos^2\phi_0\right)\right\}-H\cdot r\left\{\frac{1}{5}(\cos^5\phi_0-\cos^5\phi)\right.$$
$$\left.+\frac{1}{4}(\cos^4\phi-\cos^4\phi_0)\cos\phi_0\right\}-G.r\left\{\frac{1}{4}(\cos^4\phi_1-\cos^4\phi)\sin\phi_1\right.$$
$$\left.\left.-\frac{1}{5}\sin^3\phi\left(\frac{2}{3}+\cos^2\phi\right)+\frac{1}{5}\sin^3\phi_1\left(\frac{2}{3}+\cos^2\phi_1\right)\right\}\right]$$

Applying these formulæ to the right springing point by substituting $\phi=-\phi_0$, $x=l$, $y=0$, and, for yielding abutments, putting their relative rotation $\Delta\phi-\Delta\phi_0=c_1$ and their relative displacements $\Delta x-\Delta x_0=c_2=\Delta$ and $\Delta y-\Delta y_0=c_3$ we obtain, by solving for the unknowns,

$$\left.\begin{aligned}H=&\frac{(a_1\beta-a_0\beta_1)\sin\phi_1+\frac{2}{5}(\cos^5\phi_0-\cos^5\phi_1)a_0-\frac{1}{4}(\cos^4\phi_0-\cos^4\phi_1)\beta_0}{4a_0\gamma_0-\beta_0^2}.G\\&-\frac{2a_0(c_2+c_1\cdot r\cdot\cos\phi_0)+c_1\cdot r\cdot\beta_0+2\omega t r\phi_0\beta_0}{r(4a_0\gamma_0-\beta_0^2)}\cdot\frac{EI_0}{r^2}\end{aligned}\right\}\text{(516.}$$

$$\left.\begin{aligned}V_1=\mathbf{V}_1+X_1=\mathbf{V}_1&+\frac{5}{8}\frac{\gamma_1\cdot\sin\phi_0-\gamma_0\cdot\sin\phi_1}{\sin^4\phi_0\left(\frac{2}{3}+\cos^2\phi_0\right)}\cdot G\\&+\frac{5}{2}\frac{EI_0}{r^3}\frac{l\cdot\Delta\phi_0-c_1\frac{l}{2}-c_3}{\sin^3\phi_0\left(\frac{2}{3}+\cos^2\phi_0\right)}\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\text{(517.}$$

with which the end moments are obtained:

$$\left.\begin{aligned}M_1=&\left[-X_1.\sin\phi_0+H\frac{\beta_0-2a_0\cos\phi_0}{2a_0}-G\cdot\frac{1}{2a_0}\left(\frac{1}{4}(\cos^4\phi_0\right.\right.\\&\left.\left.-\cos^4\phi_1)+a_0\sin\phi_0-a_1\sin\phi_1\right)\right]r+\frac{EI}{r}\frac{c_1+2\omega t\phi_0}{2a_0}\end{aligned}\right\}\ldots\text{(518.}$$

With the quantities H, V_1 and M_1, we also have the effective forces, i. e., the axial thrust and bending moment, at any section of the arch.

The calculation of the above expressions is facilitated by the following table:

ϕ Degrees	$arc\,\phi$	$sin\,\phi$	$cos\,\phi$	$sin^2\phi$	$cos^2\phi$	$cos^4\phi$	cos	$\alpha - \delta\,sin\,\phi$	β	β^2	γ
5	0.08727	0.08716	0.99619	0.00760	0.99240	0.98486	0.98112	0.08694	0.17395	0.03024	0.08671
10	0.17453	0.17365	0.98481	0.03015	0.96985	0.94060	0.92631	0.17191	0.34208	0.11703	0.17019
15	0.26180	0.25882	0.96593	0.06699	0.93301	0.87051	0.84085	0.25304	0.50047	0.25047	0.24749
20	0.34907	0.34202	0.93969	0.11698	0.88302	0.77972	0.73270	0.32868	0.64475	0.41571	0.31628
25	0.43633	0.42262	0.90631	0.17861	0.82139	0.67469	0.61148	0.39746	0.77182	0.59571	0.37499
30	0.52360	0.50000	0.86603	0.25000	0.75000	0.56250	0.48714	0.45833	0.87983	0.77408	0.42292
35	0.61087	0.57358	0.81915	0.32899	0.67101	0.45035	0.36883	0.51068	0.96817	0.93736	0.46020
40	0.69813	0.64279	0.76604	0.41318	0.58682	0.34446	0.26380	0.55427	1.03737	1.07620	0.48768
45	0.78540	0.70711	0.70711	0.50000	0.50000	0.25000	0.17678	0.58926	1.08905	1.18612	0.50675
50	0.87267	0.76604	0.64279	0.58682	0.41318	0.17088	0.10973	0.61620	1.12553	1.26683	0.51914
55	0.95993	0.81915	0.57358	0.67101	0.32899	0.10823	0.06208	0.63593	1.14962	1.32158	0.52647
60	1.04720	0.86603	0.50000	0.75000	0.25000	0.06250	0.03125	0.64952	1.16428	1.35559	0.53044
65	1.13446	0.90631	0.42262	0.82139	0.17861	0.03190	0.01348	0.65817	1.17232	1.37433	0.53231

Usually, even with circular arches, it is deemed satisfactory to make the calculations by the simpler formulæ of the parabolic arch of constant moment of inertia. In the following example, taking a masonry arch of considerable rise-ratio, the results of the exact formulæ (516) to (518) and those of the formulæ (331), (326) and (334) for a parabolic arch are compared. The latter, transformed into polar coordinates, are

$$\left\{\begin{aligned} H &= \frac{15}{64}\cdot\frac{l}{f'}\cdot\frac{(sin^2\phi_0 - sin^2\phi_1)^2}{sin^4\phi_0}\cdot G \\ X_1 &= \frac{(sin^2\phi_0 - sin^2\phi_1).\,sin\,\phi_1}{4\,sin^3\phi_0}\cdot G \\ M_1 &= -\frac{2}{3}\,(1 - cos\,\phi_0)\left[\frac{4}{5}\,\frac{f'}{f}\,\frac{sin\,\phi_0}{sin\,\phi_0 - sin\,\phi_1} - 1\right]\cdot H\cdot r. \end{aligned}\right.$$

Example. Concrete arch with circular axis of radius $r = 25$ meters. Span $l = 38.80$ m., rise $f = 8.930$ m., crown thickness $d_0 = 1.0$ m., thickness at springing $d_1 = 1.556$ m. We have $\phi_0 = 50^\circ$, $\delta = \frac{I_0}{A_0 r^2} = 0.00013$. The calculation yields the following results:

ϕ_1 Degrees	H by eq. (516)	H for the parabolic arch	X_1 by eq. (517)	X_1 for the parabolic arch	M_1 by eq. (518)	M_1 for the parabolic arch
40	0.0744	0.0870	+0.0670	0.0621	—0.09093	—0.08347
30	0.3095	0.3273	0.1082	0.0937	—0.11685	—0.10372
20	0.6476	0.6369	0.1088	0.0894	—0.07658	—0.07011
10	0.9479	0.8940	0.0676	0.0537	+0.00181	—0.00994
5	1.0354	0.9680	0.0356	0.0231	+0.04087	+0.02004
0	1.0683	0.9935	0	0	+0.07040	+0.04509
—5			—0.0356	—0.0231	+0.09541	+0.06300
—10			—0.0676	—0.0537	+0.10538	+0.07242
—20			—0.1088	—0.0894	+0.09011	+0.06680
—30			—0.1082	—0.0937	+0.04892	+0.04233
—40			—0.0670	—0.0621	+0.01261	+0.00910
	.G		.G		.G.r	

According to these results, the error in the horizontal thrust H from applying the formulæ for the parabolic arch, even for this arch of large

rise-ratio, is not very great; the error in the moments, on the other hand, is more considerable. Fig. 2 (page 286) gives a drawing of the influence lines calculated by both sets of formulæ.

Disturbing Influences. In a hingeless arch, if only the action of the loading is taken into consideration, it is assumed that there are no stresses in the arch when considered without weight. In reality, however, this perfect condition is not to be found in an arch; on the contrary, disturbing influences always appear which modify the external reactions and consequently the position and form of the line of pressures. These may arise, neglecting peculiar defects of construction, from three special causes, namely:

1. A settlement of the falsework during the erection of the arch masonry;
2. A yielding, displacement or rotation of the abutments, and
3. Changes of temperature.

The effect of the disturbances (1) and (2) takes the form of a rotation $\Delta\phi_0$ of the end joints and a horizontal displacement Δl of the points of support. If, before the arch is completed but after the partial hardening of the mortar joints (or setting of the concrete), there occurs a settlement of the falsework amounting to s at the crown, there will arise a bending moment at the ends which may attain, as a maximum, the value $M_1 = -\frac{16\,EIs}{l^2}$. The effect of a displacement of the abutments of $\Delta l = c_2$, as well as that of temperature variation, is to be calculated by the principles of § 20 or by the more exact formulæ (516) to (518).

By an application of the analytical method (a), page 282, the stresses at the crown section due to variation of temperature may also be obtained from

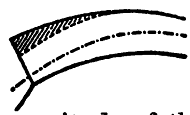
$$\left.\begin{aligned} H_t &= \frac{3}{2}\,\frac{E\,wt\,l}{\Delta s}\cdot\frac{\mathbf{d}}{\mathbf{c}^2-\mathbf{b\,i}} \\ M_0 &= -\frac{\mathbf{c}}{\mathbf{d}}\cdot H_t \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (519.$$

The coefficients **b**, **c** and **d** are determined by the equations (513). Δs is the length of an arch segment between two sections.

As a result of these disturbing influences, there may arise at certain points of the arch, particularly at the ends, tensile stresses which exceed the strength of the mortar and cause the formation of cracks. By giving the arch the necessary form and thickness, and by appropriate precautions in the erection, these undesirable effects should be prevented; but, if they do appear, the effective cross-section of the arch at those points must be correspondingly reduced (Fig. 3), and this fact

must be taken into consideration in the execution of the design. The position of the line of pressure, however, will not be materially altered by this circumstance.

Fig. 3.



With regard to the magnitude of the coefficient of elasticity E for masonry, the reader is referred to the remarks in Chap. II, fourth edition, of the "Handbuch des Brückenbaues."

The methods of design and formulæ given above may also be applied directly to arches of *reinforced concrete*. For these we have only to replace A and I in the design by their equivalent values which are obtained, in the familiar manner, by adding to the cross-section of the concrete that of the steel multiplied by $n = \frac{Es}{Ec}$.

Remarks on the Temperature Variation to be Assumed in Steel and Masonry Bridges.

In evaluating the stresses caused by change of temperature in statically indeterminate structures, we are accustomed under our climatic conditions to assume a range of temperatures from $-25°$ to $+45°$ C., so that with a mean temperature of $+10°$ C. ($= 50°$ F.), the thermal variations in metallic bridges amount to $\pm 35°$ C. ($= \pm 63°$ F.).

In this assumption the fact is duly taken into account that the *average* temperature of a steel structure exposed to the direct rays of the sun is about 10° to 15° higher than that of the air in the shade. If the structure, however, is shielded in any portions from the direct solar rays, the resulting unequal heating must be considered. On the magnitude of these differences of temperature, thorough observations were taken on the occasion of building two arch bridges at Lyons in 1886 and 1887.* For this purpose, a section 1 meter long of a plate arch having a box cross-section 900 mm. high and 800 mm. wide was used. In this test piece thermometers were inserted at eight points in holes filled with mercury and their readings taken several times a day for seven summer months. The test piece was provided with three coats of brown oil paint, closed at the end with boards and left resting freely with one long side turned toward the south. It was found that the difference of temperature between the warmest and coldest points of the

*Ann. des ponts et chaussées. 1893, II., p. 438.

steel on sunny days attained a value of 14° C. (= 25° F.) and that the average temperature of the steel rose to a value 15° C. (= 27° F.) above the temperature of the air in the shade. Surrounding the piece with a box of sheet iron effected a lowering of the maximum temperature by 7° and similarly it was found that a coating of light blue paint reduced the average temperature by about 5° C. (= 9° F.).

A temperature variation of ± 35° C. (= ± 63° F.) in the steel parts directly exposed to the sun and one of + 20° and — 35° C. (= + 36° and — 63° F.) in the parts shielded from the sun's rays may therefore be properly assumed provided a temperature of +10° C. (= 50° F.) is taken for the initial unstressed condition. But even if the temperature has some other value at the moment that the structure is released to carry its own weight, it is possible to bring about such an initial condition of stress at the erection of statically indeterminate structures as to virtually fulfil the above assumption.

In masonry bridges there cannot occur such large variations in temperature as in metallic structures. Although definite observations on this point are lacking, nevertheless it is fair to assume that massive masonry cannot attain the maximum temperature of the air in summer nor, on the other hand, can it cool down to the lowest air temperature even during prolonged frosts. If we therefore base our calculations of the temperature stresses in masonry arch bridges on a range of temperature of ± 20° C. (= ± 36° F.), all the demands of safety will be well met and, in the case of more massive masonry arches that are covered with earth, this range may even be reduced.

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